A few challenging problems

1) Find all functions f such that f' is continuous and

$$[f(x)]^2 = 100 + \int_0^x \{[f(t)]^2 + [f'(t)]^2\}dt$$

for all real x.

2) Let f be a function with the property that $f(0) = 1$, $f'(0) = 1$, and $f(a+b) = f(a)f(b)$ for all real numbers a and b. Find f.

3) Find all functions that satisfy the equation

$$\left(\int f(x)dx\right)\left(\int \frac{dx}{f(x)}\right) = -1$$

4) Find the curve that passes through the point $(3, 2)$ and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected at P.

5) Let f be a positive real-valued differentiable function. Let $f'(x) > f(x)$ for all x. For what integers k must there exist an integer N such that $f(x) > e^{kx}$ for all $x > N$.

6) Let f be a twice-differentiable function that satisfies

$$f(x) + f''(x) = -xg(x)f'(x)$$

where $g(x) > 0$ for all x. Prove that $|f(x)|$ is bounded.

7) Let f be a real-valued function with a continuous third derivative such that $f(x), f'(x), f''(x), f'''(x)$ are positive for all x. Suppose that $f'''(x) \leq f(x)$ for all x. Show that $f'(x) < 2f(x)$ for all x.