1) Provide an example or state that no such example exists (3 points each).

a) A subset of $\mathbb{R}$ that is open and compact.

Example: the empty set.

b) A subset of $\mathbb{R}$ that is not bounded and has a supremum.

Example: the interval $(-\infty, 0)$ is not bounded and has a supremum.

c) An infinite set with no accumulation points.

Example: the set of integers $\mathbb{Z}$ is infinite and has no accumulation points.

d) A monotone sequence that is bounded above and does not converge.

Example: the sequence $(s_n) = (-1, -2, -3, \dots)$ is monotone, bounded above, and not convergent.

e) A function $f : D \to \mathbb{R}$, with $D = \{0\}$, such that $\lim_{x\to 0} f(x)$ exists.

No such example exists.

2) Prove or provide a counterexample (3 points each).

a) If sequences $(s_n + t_n)$ and (t_n) both converge, then the sequence (s_n) also converges.

Proof. Suppose that $(s_n + t_n)$ converges to a and (t_n) converges to b. Let $\epsilon > 0$ be given. There exists a $N_1 \in \mathbb{R}$ such that when $n > N_1$, $|s_n + t_n - a| < \frac{\epsilon}{2}$ and there exists a $N_2 \in \mathbb{R}$ such that when $n > N_2$, $|t_n - b| < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$.

Now when $n > N$, $|s_n - (a-b)| = |s_n + t_n - a + b - t_n| \le |s_n + t_n - a| + |t_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This shows that (s_n) converges to $a - b$. □

Remark. There is another proof that utilizes the theorems about sums and constant multiples of convergent sequences. It is known that sums of convergent sequences converge. Furthermore, if a sequence converges, then that sequence multiplied by the constant -1 also converges (see [1] for these results). Combining these two results, we obtain that the difference of convergent sequences is also convergent. Now, since $s_n = (s_n + t_n) - t_n$ for all $n \in \mathbb{N}$, (s_n) converges.

b) If the sequence (s_n) diverges, then every subsequence of (s_n) must also diverge.

Counterexample. Let $s_n = (-1)^n$ for $n \in \mathbb{N}$. This sequence diverges but (s_{2n}) is a constant, and hence convergent, subsequence.

c) Let (s_n) be a convergent sequence and let (s_{n_k}) and (s_{m_k}) be two subsequences of (s_n). Then the sequence $(s_{n_k} s_{m_k})$ also converges.

Proof. If (s_n) converges, the subsequences (s_{n_k}) and (s_{m_k}) must converge, and then $(s_{n_k} s_{m_k})$ also converges. The text [1] is a good reference for these results. □

d) Let (s_n) and (t_n) be cauchy sequences. Then the sequence $(s_n t_n)$ is also cauchy.

Proof. If (s_n) and (t_n) are cauchy sequences, they must converge, and then $(s_n t_n)$ also converges. Since convergent sequences are cauchy, $(s_n t_n)$ is cauchy. Again, [1] may be consulted for references to these results. □

e) Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two functions and let c be an accumulation point of D. If $\lim_{x\to c}(fg)(x)$ and $\lim_{x\to c} g(x)$ exist, then $\lim_{x\to c} f(x)$ also exists.

Counterexample. Let $f : (0,1) \to \mathbb{R}$ and $g : (0,1) \to \mathbb{R}$ be defined by $f(x) = 1/x$ and $g(x) = x$ respectively. In this case, the function $(fg)(x) = 1$, $\lim_{x\to 0}(fg)(x) = 1$, $\lim_{x\to 0} g(x) = 0$, and $\lim_{x\to 0} f(x)$ does not exist.

3) Provide solutions for *exactly two* of the following (10 points each).

a) Prove that the set $S = \{\sqrt{2} + q : q \in \mathbb{Q}\}$ is dense in $\mathbb{R}$, that is, for all $x, y \in \mathbb{R}$ that satisfy $x < y$, there exists a $s \in S$ so that $x < s < y$.

Proof. Let $x, y \in \mathbb{R}$, with $x < y$. It follows that $x - \sqrt{2} < y - \sqrt{2}$. Since the rationals are dense, there exists a $q \in \mathbb{Q}$ such that $x - \sqrt{2} < q < y - \sqrt{2}$. This implies that $x < \sqrt{2} + q < y$. Note that $\sqrt{2} + q \in S$. This establishes the density of S. □

b) Let $s_1 = 1$ and let $s_{n+1} = \sqrt{1 + 2s_n}$, for $n \geq 1$. Show that the limit of this sequence exists and then find it.

Proof. First we provide a short argument to show that this sequence is bounded above. Observe that $s_1 = 1 \leq 3$. Now suppose that $s_k \leq 3$ for some $k \in \mathbb{N}$. So $s_{k+1} = \sqrt{1 + 2s_k} \leq \sqrt{1 + 2(3)} = \sqrt{7} \leq 3$. This induction argument shows that this sequence is bounded above by 3.

Now we will use another induction argument to prove that this is an increasing sequence. Observe that $s_1 = 1 < s_2 = \sqrt{3}$. Suppose that $s_k < s_{k+1}$ for some $k \in \mathbb{N}$. Now $s_{k+1} = \sqrt{1 + 2s_k} < \sqrt{1 + 2s_{k+1}} = s_{k+2}$. This establishes that the sequence is increasing.

Every increasing sequence that is bounded above converges; so let $\lim s_n = s$. Since this limit exists, we may conclude that $\lim(s_{n+1}s_{n+1}) = \lim(1 + 2s_n)$. Now, several limit theorems (see [1] for instance) imply that $s^2 = 1 + 2s$, and so, $s = 1 \pm \sqrt{2}$. Since the sequence is increasing, it is bounded below by $s_1 = 1$. It now follows that $s = 1 + \sqrt{2}$. □

c) Show that $\lim_{x\to 0} x^2/|x| = 0$.

Proof. Let $\epsilon > 0$ be given and then let $\delta = \epsilon$. Observe that $\frac{x^2}{|x|} = |x|$ if $x \neq 0$. So when $0 < |x - 0| < \delta = \epsilon$, we have $\left|\frac{x^2}{|x|} - 0\right| = |x| < \delta = \epsilon$. This proves that $\lim_{x\to 0} x^2/|x| = 0$. □

d) Let (s_n) be a sequence of real numbers. If the subsequence (s_{2n}) converges to $s \in \mathbb{R}$ and the subsequence (s_{2n-1}) converges to s, then the sequence (s_n) also converges to s.

Proof. Let $\epsilon > 0$ be given. Since (s_{2n}) converges to s, there exists a $N_1 \in \mathbb{R}$ so that when $n > N_1$, $|s_{2n} - s| < \epsilon$. Since (s_{2n-1}) converges to s, there exists a $N_2 \in \mathbb{R}$ so that when $n > N_2$, $|s_{2n-1} - s| < \epsilon$. Let $N = \max\{N_1, N_2\}$. Now when $n > 2N$, $|s_n - s|$ is either equal to $|s_{2n'} - s|$ or $|s_{2n'-1} - s|$, where $n' > N$. In both cases, $|s_n - s| < \epsilon$. This shows that (s_n) converges to s. □

References

[1] S. Lay, *Analysis with an introduction to proof*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1986.