1) Provide an example or state that no such example exists. Solve *exactly five* (3 points each).

a) A function $f : (a, b) \to \mathbb{R}$ that is differentiable on the interval (a, b) but not uniformly continuous on (a, b).

Example: $f(x) = \frac{1}{x}$ is differentiable on $(0, 1)$ but not uniformly continuous on $(0, 1)$.

b) A function $f : (a, b) \to \mathbb{R}$ that is uniformly continuous on (a, b) but not differentiable on (a, b).

Example: $f(x) = |x|$ is not differentiable on $(-1, 1)$ and it is uniformly continuous on any interval.

c) A continuous function $f : \mathbb{R} \to \mathbb{R}$ and a closed subset G of $\mathbb{R}$ such that $f(G)$ is not closed.

Example: when $f(x) = e^x$ and $G = \mathbb{R}$ (a closed set), $f(G) = (0, \infty)$, which is open and not closed.

d) A continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open subset H of $\mathbb{R}$ such that $f(H)$ is not open.

Example: when $f(x) = 0$ for all $x \in \mathbb{R}$ and $H = \mathbb{R}$ (an open set), $f(H) = \{0\}$, which is closed and not open.

e) A one-to-one (injective) function $f : \mathbb{R} \to \mathbb{R}$ that is continuous on $\mathbb{R}$ and an open set $G \subset \mathbb{R}$ such that $f(G)$ is not open.

No such example exists. See exercise 22.11 in [1].

f) A function $f : [a, b] \to \mathbb{R}$ that is differentiable on $[a, b]$ but not Riemann integrable on $[a, b]$.

No such example exists since differentiability on $[a, b]$ implies continuity on $[a, b]$ and continuous functions on closed and bounded intervals are always integrable.

g) A function $f : [a, b] \to \mathbb{R}$ that is differentiable on $[a, b]$ and f' is not integrable on $[a, b]$.

Example: let $f : [-1, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin\left(\frac{1}{x^2}\right) & \text{if } x \neq 0 \end{cases}$$

Now f' is unbounded on $[-1, 1]$, and hence, not integrable.

2) Prove or provide a counterexample. Solve *exactly five* (3 points each).

a) If $f : (a, b) \to \mathbb{R}$ is differentiable on (a, b) and f is strictly decreasing on (a, b), then $f'(x) < 0$ for all $x \in (a, b)$.

Counterexample: $f(x) = -x^3$ is strictly decreasing and differentiable on $(-1, 1)$ and $f'(0) = 0$.

b) Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be two real-valued functions defined on the domain $D \subset \mathbb{R}$. If f and g are both uniformly continuous on D, then $f - g$ is also uniformly continuous on D.

Proof. Let $\epsilon > 0$ be given. There exists a $\delta_1 > 0$ such that when $x, y \in D$ and $|x - y| < \delta_1$, $|f(x) - f(y)| < \epsilon/2$. Also there exists a $\delta_2 > 0$ such that when $x, y \in D$ and $|x - y| < \delta_2$, $|g(x) - g(y)| < \epsilon/2$. Now let $\delta = \min\{\delta_1, \delta_2\}$ and observe that when $|x - y| < \delta$, $|(f - g)(x) -$

$(f-g)(y)| \le |f(x)-f(y)|+|g(x)-g(y)| < \epsilon/2+\epsilon/2 = \epsilon$. This establishes uniform continuity of $f-g$ on D. □

c) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function on $\mathbb{R}$, then f assumes its maximum value or its minimum value.

Counterexample: $f(x) = \tan^{-1} x$ is continuous and bounded on $\mathbb{R}$ with the property that the maximum and minimum values are both not attained/assumed; $\tan^{-1} x$ is an increasing function bounded between its horizontal asymptotes $y = \pm\pi/2$.

d) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$. Then for every $c \in \mathbb{R}$, there exist $a, b \in \mathbb{R}$ such that $a < c < b$ and $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Counterexample: when $f(x) = x^3$, $f'(0) = 0$ and for all $a < 0 < b$, $\frac{f(b)-f(a)}{b-a} > 0$ because the function is strictly increasing.

e) Let $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ be two functions and let $c \in (a, b)$. If fg is differentiable at c and g is differentiable at c, then f is also differentiable at c.

Counterexample: let $f(x) = |x|$ and let $g(x) = x$, and then observe that $x|x|$ and x are both differentiable at 0 but $|x|$ is not differentiable at zero.

f) Let $f : [a, b] \to \mathbb{R}$ be bounded on $[a, b]$. If f^2 is integrable on $[a, b]$, then so is f.

Counterexample: let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Now observe that f^2 is a constant function, and hence, integrable. However, f is not integrable because $L(f, P) = -1$ and $U(f, P) = 1$ for any partition of P, which implies that $L(f) = -1$ and $U(f) = 1$.

g) Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. Then there exists a $c \in [a, b]$ such that $f(c)(b-a) = \int_a^b f\,dx$.

Counterexample: let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Let $P_n = \{-1, -1+\frac{1}{n}, -1+\frac{2}{n}, \dots, 0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$ and notice that $U(f, P_n) = \frac{2}{n}, L(f, P_n) = 0$. This implies that $U(f) \le 0$ and $L(f) \ge 0$. Since $U(f) \ge L(f)$, we have $U(f) = L(f) = 0$, that is, $\int_{-1}^{1} f\,dx = 0$.

In our present context, $f(c)(b-a) = \int_a^b f\,dx$ is the equation $2f(c) = \int_{-1}^{1} f\,dx = 0$. So a solution exists if and only if there is a $c \in [a, b]$ such that $f(c) = 0$. Note that no such point exists.

3) Provide solutions for *exactly two* of the following (10 points each).

a) Let $f : (a, b) \to \mathbb{R}$ be a function that satisfies $|f(x)-f(y)| \le (x-y)^2$ for all $x, y \in (a, b)$. Prove that f is constant on (a, b).

Proof. Let c be a point in (a,b) and note that $|f(x)-f(c)| \leq (x-c)^2$ for all $x \in (a,b)$. This implies that for all $x \in (a,b)$ and $x \neq c$,

$$-\left|\frac{(x-c)^2}{x-c}\right| \leq \frac{f(x)-f(c)}{x-c} \leq \left|\frac{(x-c)^2}{x-c}\right|.$$

Now since

$$\lim_{x\to c} -\left|\frac{(x-c)^2}{x-c}\right| = \lim_{x\to c}\left|\frac{(x-c)^2}{x-c}\right| = 0,$$

$$f'(c) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = 0.$$

So f' is identically equal to zero on (a,b), which implies that f is constant on (a,b). □

b) Show that $|\log x - \log y| \leq |x-y|$ for all $x \geq 1$ and all $y \geq 1$. Use this inequality to prove that $\log x$ is uniformly continuous on $[1,\infty)$. Also, show that $\log x$ is not uniformly continuous on $(0,1]$.

Proof. Let $f(x) = \log x$ and assume that $x, y \in [1,\infty)$, with $x \neq y$. The Mean Value Theorem implies that there exists a $c \in (x,y)$ such that $\frac{f(x)-f(y)}{x-y} = f'(c) = \frac{1}{c}$. Notice that when $c \geq 1$, $0 \leq \frac{1}{c} \leq 1$. So $|f(x)-f(y)| = \frac{1}{c}|x-y| \leq |x-y|$ for all $x,y \in [1,\infty)$, with $x \neq y$. Also, this inequality holds trivially when $x = y$. This establishes that $|\log x - \log y| \leq |x-y|$ for all $x \geq 1$ and all $y \geq 1$.

Now let $\epsilon > 0$ be given and then let $\delta = \epsilon$. So when $x,y \in [1,\infty)$ and $|x-y| < \delta = \epsilon$, we have $|\log x - \log y| \leq |x-y| < \epsilon$. This proves that $\log x$ is uniformly continuous on $[1,\infty)$.

Consider the cauchy sequence (s_n) in $(0,1]$ defined by the equation $s_n = \frac{1}{n}$ for $n \in \mathbb{N}$. It suffices to show that $(f(s_n)) = (-\log n)$ is not cauchy in order to prove that $\log x$ is not uniformly continuous on $(0,1]$. We will show that $(-\log n)$ is not cauchy by proving that $(-\log n)$ is not bounded. Let $M > 0$ be given and then choose $N \in \mathbb{N}$ such that $N > e^M$. Observe that $|-\log N| > \log e^M = M$—this follows from the fact that $\log x$ is an increasing function ($f'(x) = \frac{1}{x} > 0$ for $x > 0$). So $(-\log n)$ is not bounded, and therefore, $\log x$ is not uniformly continuous on $(0,1]$. □

c) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows.

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ -x^2 & \text{if } x \text{ is irrational} \end{cases}$$

Show that f is continuous only at 0. Is f differentiable anywhere? Explain.

Proof. Suppose f is continuous at some $x \in \mathbb{R}$. Let (x_n) be a sequence of rationals converging to x and let (y_n) be a sequence of irrationals converging to x. So $\lim f(x_n) = x^2$ and $\lim f(y_n) = -x^2$. Since f is continuous at x, $x^2 = -x^2$, which is only possible at $x = 0$. So f is not continuous when $x \neq 0$. This also means that f is not differentiable when $x \neq 0$.

We will now show that $f'(0) = 0$. For all $x \neq 0$, we have

$$-\left|\frac{x^2}{x}\right| \leq \frac{f(x) - f(0)}{x - 0} \leq \left|\frac{x^2}{x}\right|.$$

Now since

$$\lim_{x \to 0} -\left|\frac{x^2}{x}\right| = \lim_{x \to 0}\left|\frac{x^2}{x}\right| = 0,$$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

This also proves that f is continuous at 0. □

d) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $\int_a^b (f(x))^2\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof. Let $g(x) = (f(x))^2$ and observe that g is continuous on $[a, b]$, $g(x) \geq 0$ on $[a, b]$, and $L(g) = 0$ (because $\int_a^b g\,dx = 0$). Now exercise 29.7 in [1] implies that g is identically zero on $[a, b]$. Since $g(x) = (f(x))^2$, f must be identically equal to zero on $[a, b]$. □

e) If $f : [0, 1] \to [0, 2]$ is a differentiable on $[0, 1]$, with $f(0) = 0$ and $f(1) = 2$, then there exists a point $c \in [0, 1]$ such that $f'(c) = 2$. You may assume that f' is continuous on $[0, 1]$ to simplify the proof; however, this result holds even when f' is not a continuous function.

Proof. The simplest approach is to apply the Mean Value Theorem. There exists a $c \in (0, 1)$ such that $f'(c) = \frac{f(1)-f(0)}{1-0} = 2$. There are at least two other proofs that do not (directly) utilize the Mean Value Theorem; these are outlined below.

If there exist points $x, y \in [0, 1]$ such that $f'(x) < 2$ and $f'(y) > 2$, then the Intermediate Value Theorem for Derivatives would imply that there exists a point $c \in [0, 1]$ such that $f'(c) = 2$. If we assume that f' is continuous on $[0, 1]$, the Intermediate Value Theorem (for continuous functions) is sufficient and we need not employ the Intermediate Value Theorem for Derivatives.

So assume that either $f'(x) < 2$ for all $x \in [0, 1]$ or $f'(x) > 2$ for all $x \in [0, 1]$. Now if f' is continuous on $[0, 1]$, f' is integrable on $[0, 1]$, and we may apply the Fundamental Theorem of Calculus to conclude that $2 = f(1) - f(0) = \int_0^1 f'(x)\,dx$. However, if $f'(x) < 2$ for all $x \in [0, 1]$, we have $\int_0^1 f'(x)\,dx < \int_0^1 2\,dx = 2$. Similarly, if $f'(x) > 2$ for all $x \in [0, 1]$, we have $\int_0^1 f'(x)\,dx > \int_0^1 2\,dx = 2$. In both cases we contradict our earlier conclusion that $\int_0^1 f'(x)\,dx = 2$. This completes the proof when f' is continuous on $[0, 1]$.

Now suppose that f' is not continuous on $[0, 1]$. We can still assume that either $f'(x) < 2$ for all $x \in [0, 1]$ or $f'(x) > 2$ for all $x \in [0, 1]$; otherwise the Intermediate Value Theorem for derivatives would imply that there exists a point $c \in [0, 1]$ such that $f'(c) = 2$. If $f'(x) < 2$ for all $x \in [0, 1]$, let $h(x) = f(x) - 2x$. Observe that $h'(x) = f'(x) - 2 < 0$ for all $x \in [0, 1]$ and $h(0) = 0$. This implies that the function is strictly decreasing on $[0, 1]$ and $f(x) < 2x$ for all $x \in (0, 1]$. In particular, $f(1) < 2$, which contradicts the fact that $f(1) = 2$. Similarly, if $f'(x) > 2$ for all $x \in [0, 1]$, we argue that $f(x) > 2x$ for all $x \in (0, 1]$. In particular, $f(1) > 2$, which is again a contradiction. So there exists a point $c \in [0, 1]$ such that $f'(c) = 2$. □

f) Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$and let $g : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$, with $g(x) \geq 0$ for all $x \in [a, b]$. Prove that there exists a $c \in [a, b]$ so that $\int_a^b (fg)\, dx = f(c) \int_a^b g\, dx$. This result is often referred to as the Extended Mean Value Theorem for Integrals.

Proof. Let $h(x) = \int_a^b (fg)\, dx - f(x) \int_a^b g\, dx$ for all $x \in [a, b]$; note that h is continuous on $[a, b]$ since f is continuous on $[a, b]$. Also, since f is continuous, there exist points $x_1, x_2 \in [a, b]$ where f assumes its minimum m and its maximum M respectively.

Now $f(x)g(x) \leq Mg(x)$ for all $x \in [a, b]$ since $f(x) \leq M$ and $g(x) \geq 0$ for all $x \in [a, b]$. This implies that $\int_a^b (fg)\, dx \leq \int_a^b Mg\, dx = M \int_a^b g\, dx = f(x_2) \int_a^b g\, dx$. In other words, $h(x_2) \leq 0$. Similarly, $f(x)g(x) \geq mg(x)$ for all $x \in [a, b]$ implies that $\int_a^b (fg)\, dx \geq \int_a^b mg\, dx = m \int_a^b g\, dx = f(x_1) \int_a^b g\, dx$. In other words, $h(x_1) \geq 0$. If $h(x_1) = 0$, let $c = x_1$, and if $h(x_2) = 0$, let $c = x_2$. If $h(x_1) \neq 0$ and $h(x_2) \neq 0$, the Intermediate Value Theorem applied to the continuous function h yields a point $c \in [a, b]$ such that $h(c) = 0$. In all cases, $h(c) = 0$ implies the desired equality $\int_a^b (fg)\, dx = f(c) \int_a^b g\, dx$. □

References

[1] S. Lay, *Analysis with an introduction to proof*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1986.