

## CHAPTER 1

# Basic Financial Concepts

### 1. Definitions

This is a mathematics course, in the sense that finance is used to learn new mathematical concepts and methods. Before we start, we need to have an appropriate vocabulary of financial terms. Each term has two aspects: what it means in the real world and what it means mathematically.

Let's start with some important terms commonly used in finance. Since not all financial terms are formally defined in this book. It will be helpful to have a dictionary of finance handy. *Dictionary of Finance and Investment Terms*, by John Downes and Jordan Elliot Goodman, published by Barron's, 2nd edition.

*Stock.* Ownership of a corporation represented by shares.

*Option.* A contract that gives the owner the right to buy or sell an underlying asset at a fixed price  $K$  (*strike price*) before or at a fixed time  $T$  (*expiration*). A *put* (option) is a right to sell, and a *call* is a right to buy.

*American option.* An option that can be exercised at any time prior to expiration.

*European option* An option that can be exercised only at expiration.

Note that a holder of an option never loses money and has the potential to make money. Therefore it is worth something, i.e., it has a price, which is called the *premium*. If you purchase an option (instead of being given one), the maximum loss you can have is the amount of the premium. The premium is the value of an option (or of any financial instrument). It should not be confused with *commission*, which is the fee you pay to a broker for the service provided. When purchasing an option from a broker, you need to pay both premium and commission.

To simplify the exposition, unless otherwise stated, we assume that the market is *frictionless*, i.e., we ignore commissions in all our computations. Another assumption we often use is that the market has *infinite capacity*, i.e., you are able buy from the market or sell to the market any number of shares of an underlying asset at the market price.

We will use the following notations for option prices:

European call option:  $c(t, T, K; S)$ ;

European put option:  $p(t, T, K; S)$ ;

American call option:  $C(t, T, K; S)$ ;

American put option:  $P(t, T, K; S)$ ;

where the parameters are

- $t$  = current time,
- $T$  = expiration time,
- $K$  = strike price,
- $S$  = current asset price (spot price).

A parameter can be omitted if its value is understood. We will explain these parameters in the next section.

*Interest.* The cost of using money or an asset, expressed as a rate per period of time (usually a year). The *nominal rate* is the rate without compounding, and the *effective rate* is the rate after compounding.

*Arbitrage.* Profiting from differences in price of the same currency or commodity traded on different markets. For example, if the price of gold is higher in London than in New York one can buy in New York and sell in London for a risk-free profit. That would be an arbitrage opportunity. While these opportunities might exist, it would necessarily be of short duration. Market observers attempting this strategy would drive up the price of gold in New York with increased demand and lower the price of gold in London with increased supply. This would end the arbitrage opportunity.

An *arbitrage argument* is an argument based on the principle that there are no arbitrage opportunities without risk. Prices of many financial instruments are determined by an arbitrage argument. We will discuss this topic later in great detail.

*Bond.* An interest-bearing corporate or government security with the interest and principal payable at a fixed time. It may also pay a cash dividend (coupon) at regular time intervals during the life of the contract.

A bond has a *face value* (or *par value*), a *maturity date*, and a coupon rate. For example, a bond with face value \$5,000, maturity date August 1, 2021 and coupon rate 8% will allow the bearer to collect \$200 (4%) every 6 months for the next 20 years and a maturity payment of \$5,000 on the maturity date. The current price of the bond may be \$2,050. A *zero-coupon bond* A bond that pays no coupons (cash dividend). Most of education bonds are of this type.

*Short sale.* Sale of a security not owned by the seller in hopes of taking advantage of an anticipated price decline. For the short seller, it is basically a promise to deliver the security at a fixed price on a future date. If the seller actually owns the security, his short position is said to be covered. The concept of short sales is the stock equivalent of borrowing money from a bank. Just as you can borrow cash from a bank, we also assume that you can borrow stocks from a stocker broker. But there is an important difference. When you return a cash loan, you pay back the borrowed amount (principal) and the interest for the loan period, whereas for a loan on a stock, you return in kind, namely, you return the same number of shares

of the same stock to the lender. Of course, in practice you need to pay a commission for the loan, but under our assumption of the frictionless market, the commission is ignored.

*Forward contract.* An agreement to buy or sell a specific amount of a commodity or financial instrument at the current price with delivery and settlement at a future date.

Note that a forward contract is not an option, i.e., the transaction has to take place at the specified time.

A large part of this course is devoted to option pricing theory developed by Black, Scholes and Merton. The subject was born in 1973 with the papers “The pricing of options and corporate liabilities” in *Journal of Political Economics* by Black and Scholes and “The theory of rational option pricing,” in *Bell J. Econ. Management Sci.* by Merton. In these papers, the famous Black-Scholes formula for pricing options was derived. In the same year, the Chicago Board Options Exchange (CBOE) opened. In the intervening years the Black-Scholes formula has become the accepted way to price stock and other options. There are trillions of dollars involved in options exchanges yearly. Moreover, option pricing and risk management has become a major area of applied mathematics. Research activity in Mathematical Finance requires highly sophisticated tools from mathematics (mainly stochastic analysis) and is carried on in academia as well as in the financial industry.

## 2. Option contracts

Here is a list of essential terms related to options, some of which you are already familiar with from the last section. Consult a finance dictionary and try to memorize their meanings. If you cannot find a term in your dictionary, try to guess what it means. Dual terms are grouped together.

- Call option, put option
- Strike Price
- Expiration date (expiry)
- Exercise
- American option
- European option
- Option buyer, option writer
- In the money, at the money, out of the money
- Intrinsic value
- Premium

*Intrinsic value* is the profit to be made on an option at the time of exercise. If the strike price of a call option is \$50 and current market price is \$55, the intrinsic value is \$5. If there is profit to be made, the option is *in the money*; if exercising the option will incur a loss, the option is said to be out of the money; if the strike price is equal to the market price, it is said to be *at the money*.

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Reading assignment: *Options for the Stock Investor*, by James B. Bittman, McGraw-Hill, New York (1997), Chapter 6.

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What are options? We first recall the two types of options, a call option and a put option. A holder of a call has the right to buy a specified asset at a price determined by the option contract. A holder of a put has the right to sell a specified asset at a price determined by the option contract. Options can also be classified by the way they are exercised. For an *American option*, the holder can exercise his right at any time up to time  $T$ . For a *European option*, the holder can exercise his right only at time  $T$ . Other types of options are called exotic options, including the *Asian option*, in which the price to sell or buy the stock is the average of the stock price during the life of the option instead of the market (or the spot) price.

The holder of an option contract is said to have a *long position* on the option, meaning that he has purchased the option from an agent and has paid an premium for it. In this case, the agent is said to hold a *short position*. The asset specified in the option contract is called the *underlying asset*, or the *underlying* for short.

The phrase “underlying asset” is a general and loose term to denote the commodity specified in a contract. It usually conveys the idea that the future price of this commodity cannot be predicted with certainty. But mathematically it does not have to refer to something tangible. A contract doesn't have to be based on a tangible asset at all. For example, it can be based on the S&P 500 index, or the temperature at noon on Wall Street on June 1, 2005, or anything on which people have different opinion.

Two parameters in an options contract are its terminal or expiration time  $T$ , sometimes also referred to as *expiry* or maturity, and the strike price  $K$ , which is the agreed sale price in the option contract. To determine the price of an option we need also need know at least the current time  $t$  and the current stock price  $S_t$ .

Here are some very practical questions to consider at this point.

- Why would an agent enter an options contract?
- What does an agent have to gain from entering an options contract?
- What should an agent pay for an options contract?

There are two reasons agents enter into an options contract. The first reason is that options can be used as a hedge against risk. These agents, called hedgers, routinely face the problem of acquisition of goods, foreign currency or loans. These goods may be necessary in the normal course of their business activities. The fluctuations in price, exchange rate, interest rate with time are random, or at the very least unpredictable. An options contract can act as insurance against unfavorable price fluctuations.

A second reason for entering an options contract is to gamble on the price fluctuation of stocks for possible significant profits. For this the agent

(a speculator) has to be willing to accept risk. A hedger in effect transfers (or sells) his risk to a speculator. Hedging by purchase of an options contract is tantamount to buying insurance, which is not free. The advantage of a call option over outright purchase of the stock is that with a call option the loss is capped at the premium of the option (which would be the loss in the long position of an unexercised option), whereas with the stock the loss can be the whole purchase price of the stock.

Let's assume that we have purchased a European call option at time  $t = 0$ . We denote its value at time  $t$  by  $c(t, T, K, S_t)$ . Thus  $c(0, T, K, S_0)$  is the premium paid at  $t = 0$  for the option. The profit from this option at the expiration is easy to write down. First we introduce two functions. We use  $x^+$  to denote 0 if  $x < 0$  and  $x$  otherwise. This function is called the positive part of  $x$ . The negative part of  $x$  is the function  $x^-$ , which has the value 0 if  $x > 0$  and  $-x$  otherwise. Note that these two functions satisfy the relations.

$$x = x^+ - x^-, \quad |x| = x^+ + x^-.$$

At time  $T$ , if the option is in the money, meaning  $S_T \geq K$ , we exercise the option and have the profit  $S_T - K$ ; otherwise we let the option expire and the profit in this case is 0. It follows that the value of the option at time  $T$  is just  $(S_T - K)^+$ . This can be written as

$$c(T, T, K, S_T) = (S_T - K)^+.$$

A similar argument can be applied to a put option:

$$p(T, T, K, S_T) = (K - S_T)^+.$$

Of course, the cost of purchasing the contract (premium plus commission) must be subtracted to obtain the net profit.

We have seen that the function  $c(t, T, K, S)$  has a terminal value

$$c(T, T, K, S) = (S - K)^+.$$

What we are interested in is its initial value  $c(0, T, K, S)$ . After all we need to determine the fair price for these instruments. *One of our goals in this course is to develop a theory which enable us to determine this function completely.* But even without knowing this function explicitly, we can use some common sense (and some not so common sense) to derive several important properties this function must satisfy. This will be done in the next few sections.

### 3. Examples of options

The value of a call option  $c(t, T, K, S)$  is a decreasing function of  $K$ , because the option is less valuable if you have to pay a higher price when you exercise it. Symbolically,

$$c(t, T, K_1, S) \geq c(t, T, K_2, S) \quad \text{if} \quad K_1 \leq K_2.$$

EXAMPLE 3.1. (Bull spread) Suppose you are bullish on a given stock, that is, you expect that its value will rise. Can you use options to profit from a rise in the stock's price? Here's what you might try:

- (a) Buy a call with strike price  $K_1$ .
- (b) Sell a call with strike price  $K_2 > K_1$ .

Suppose that the value of the call options are  $c(K_i) = c(t, T, K_i)$ ,  $i = 1, 2$ . We have just seen that

$$c(K_2) \leq c(K_1).$$

Let  $P(t)$  denotes the value of this portfolio at time  $t$ . Then

$$P(t) = c(t, T, K_1, S_t) - c(t, T, K_2, S_t).$$

At  $t = 0$ ,

$$P(0) = c(K_1) - c(K_2) \geq 0,$$

which is the cost to establish the portfolio. At expiration

$$P(T) = (S_T - K_1)^+ - (S_T - K_2)^+.$$

This function is nonnegative function of the stock price  $S_T$  (see the diagram).

$$[\text{Diagram of } f(s) = (s - K_1)^+ - (s - K_2)^+.]$$

The maximum loss of the bull spread is limited to  $c(K_1) - c(K_2)$  if the stock price decreases below  $K_1$ , and the maximum profit is limited to  $K_2 - K_1 - c(K_1) + c(K_2)$  if the stock rises above  $K_2$ .

Compare the bull spread with a straightforward call option, which can also be used to profit from a rise in the stock's price.

EXAMPLE 3.2. (Bear spread) On the other hand if you're pessimistic about the outlook for a stock, you can construct a portfolio of options to profit in case you turn out to be correct. Try reversing the procedure with the bull spread. That is

- (a) Buy a put with strike price  $K_2$ .
- (b) Sell a put with strike price  $K_1 < K_2$ .

The payoff of the option is

$$P(T) = (K_2 - S_T)^+ - (K_1 - S_T)^+.$$

$$[\text{Diagram of } f(s) = (K_2 - s)^+ - (K_1 - s)^+.]$$

The cost of establishing this portfolio is

$$P(0) = p(0, T, K_2, S_0) - p(0, T, K_1, S_0).$$

Since the payoff is nonnegative we have

$$p(0, T, K_2, S_0) \geq p(0, T, K_1, S_0),$$

Namely, the value of a put option is an increasing function of the strike price.

EXAMPLE 3.3. (Butterfly spread) Let's now discuss the so-called butterfly spread. This time the agent expects the underlying asset price will not move very much between the times 0 and  $T$ . He does the following:

- a. Buy a call with strike  $K_1$ .
- b. Buy a call with strike  $K_2 > K_1$ .
- c. sell two calls with strike  $K_3 = (K_1 + K_2)/2$ .

In this case, the payoff at expiration is

$$P(T) = (S_T - K_2)^+ + (S_T - K_1)^+ - 2 \left( S_T - \frac{K_1 + K_2}{2} \right)^+.$$

The cost of establishing this portfolio is

$$C = c(K_1) + c(K_2) - 2c(K_3).$$

The diagram of the payoff as a function of the stock price  $S_T$  is

$$[\text{Diagram of } f(s) = (s - K_2)^+ + (s - K_1)^+ - 2 \left( s - \frac{K_1 + K_2}{2} \right)^+ .]$$

Note that the payoff function is nonnegative, i.e., this portfolio can never lose money. It follows that the cost of establishing this portfolio is nonnegative, namely,  $C \geq 0$  in this case, or

$$c \left( \frac{K_1 + K_2}{2} \right) \leq \frac{c(K_1) + c(K_2)}{2}.$$

A function with this property is called a convex function.

A function  $f(x)$  on an interval  $[a, b]$  is called *convex* if for any  $a < x, y < b$ , the graph of the function on the interval  $[c, d]$  lies below the straight line joining the two points  $(x, f(x))$  and  $(y, f(y))$  on the curve. This condition can be written as follows: for any  $x, y \in [a, b]$  and any  $0 \leq \lambda \leq 1$ ,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

When  $\lambda = 1/2$ , we have

$$f \left( \frac{x + y}{2} \right) \leq \frac{f(x) + f(y)}{2}.$$

In fact, it can be shown that this special case implies that the above inequality holds for all  $0 \leq \lambda \leq 1$ . Hence we have the following conclusion.

THEOREM 3.4. *The price of a call option  $c(K) = c(0, T, K, S)$  is a decreasing, convex function of the strike price  $K$ .*

For a smooth function (with continuous second derivative), there is a simple condition for convexity.

THEOREM 3.5. *If the second derivative  $f''$  is nonnegative, then  $f$  is convex.*

If  $-f$  is a convex function, then  $f$  is called a concave function.

EXAMPLE 3.6. (Straddle) Now the agent expects a big change in the stock price, though he is uncertain in which direction. This time the agent establishes the following portfolio:

- (a) Buy a call with strike  $K$ .
- (b) Buy a put with strike  $K$ .

Then his payoff at expiration is

$$P(T) = (S_T - K)^+ + (K - S_T)^- = |S_T - K|.$$

[Diagram of  $f(s) = |s - K|$ .]

In this case the holder of the above portfolio stands to gain when the stock price makes either an up or down movement.

EXAMPLE 3.7. (Strangle) The agent has the feeling that the asset price will move well out of its present range. Accordingly he creates a portfolio composed of

- (a) A long position in a call with strike  $K_2$ .
- (b) A long position in a put with strike  $K_1 < K_2$ .

Here  $K_1 < S_0 < K_2$ , where  $S_0$  is the stock price at time  $t = 0$ . This combination resembles a straddle, but the diagram between  $K_1$  and  $K_2$  is flat. The time  $T$  payoff is

$$P(T) = (S_T - K_2)^+ + (K_1 - S_T)^+$$

[Diagram of  $f(s) = (s - K_2)^+ + (K_1 - s)^+$ .]

The payoff for a straddle with  $K = (K_1 + K_2)/2$  always exceeds the payoff for a strangle with strikes  $K_1$  and  $K_2$ . The price of a straddle must therefore be greater than the price of a strangle.

*Spreads* are strategies of buying and selling the same options at different strike prices and/or with different expiration times. They are further classified as follows:

- vertical spread*: different strike prices and same expiration time;
- horizontal spread*: same strike price and different expiration times;
- diagonal spread*: different strike prices and different expiration times.

Horizontal and diagonal spreads are collectively referred to as *calendar spreads*. Here is an example of calendar spreads.

EXAMPLE 3.8. (Calendar spread) Our agent now wants to profit in case the asset price is low at  $T_1$  and rises at a later time  $T_2$ . This calls for the following portfolio:

- (a) Sell a call with strike  $K$  and expiration  $T_1$ .
- (b) Buy a call with strike  $K$  and expiration  $T_2$ .

The resulting payoff is  $(S_{T_2} - K)^+ - (S_{T_1} - K)^+$ . It depends on the asset prices at both time  $T_1$  and  $T_2$ . If the graph of  $S_t$  is below  $K$  at  $T_1$  and above  $K$  at  $T_2$ , the agent will receive  $S_{T_2} - K$ .



#### 4. Exercises

EXERCISE 4.1. John Doe sold short 3,000 shares of XYZ stock short at \$72 per share. A month later, a false alarm caused the stock price to plunge to \$45 per share, at which John Doe bought 3,000 shares to cover his short position. He then bought 3,500 shares of the same stock at \$52, and later sold these shares at \$113 when the price quickly rebounded. How much profit did John Doe make on XYZ?

EXERCISE 4.2. Follow the proof for the European call option price function to show that the European put option price  $p(K) = p(t, T, K, S)$  is an increasing, convex function of  $K$ .

EXERCISE 4.3. (Strap) An agent is fairly confident that a stock price will move, but wishes to profit more from an upturn, which he feels is more likely than a downturn. He acquires the following portfolio:

- (a) Buy two calls with strike  $K$  and maturity  $T$ .
- (b) Buy one put with maturity  $K$  and maturity  $T$

[Graph of the function  $f(s) = 2(s - K)^+ + (K - s)^+$ .]

This gives a graph similar to that for the straddle, but the slope of the graph to the right of  $K$  for the strip is twice as large as for the straddle. If the strike price at expiration is \$120, what is the payoff of a strap with strike price \$130?

EXERCISE 4.4. (Strip) An agent has similar expectations as for the strip except he is more confident that a downturn will occur than an upturn. This agent acquires the portfolio

- (a) Buy a call with strike  $K$  and expiration  $T$ .
- (b) Buy two puts with strike  $K$  and expiration  $T$ .

The graph of the payoff for this combination is obtained from the graph of the strip by reflecting in the vertical line  $s = K$ . If the stock price at expiration is \$50, what is the payoff with a strike price \$35?

EXERCISE 4.5. Let  $K_1 < K_2 < K_3$ . Show that the three call option prices  $c(K_i) = c(0, T, K, S)$  satisfies

$$c(K_2) \leq \frac{K_3 - K_2}{K_3 - K_1} \cdot c(K_1) + \frac{K_2 - K_1}{K_3 - K_1} \cdot c(K_3).$$

If we let  $\lambda = (K_3 - K_2)/(K_3 - K_1)$ , then  $0 \leq \lambda \leq 1$  and

$$c(\lambda K_1 + (1 - \lambda)K_3) \leq \lambda c(K_1) + (1 - \lambda)c(K_3),$$

which shows directly that  $c(K)$  is a convex function of  $K$ .