Hyperbolic triangular prisms with one ideal vertex

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September 27, 2018

Abstract

In this paper, we classify all of the five-sided three-dimensional hyperbolic polyhedra with one ideal vertex, which have the shape of a triangular prism, and which give rise to a discrete reflection group. We show how to find each such polyhedron in the upper half-space model by considering lines and circles in the plane. Finally, we give matrix generators in $\mathrm{PSL}_2(\mathbb{C})$ for the orientation-preserving subgroup of each corresponding reflection group.

1 Introduction

A convex polyhedron in hyperbolic 3-space $\mathbb{H}^3$ generates a discrete group of isometries if the dihedral angles at which its bounding planes meet are all integer submultiples of π – that is, each angle is of the form π/n radians, for an integer $n \geq 2$ – and if the dihedral angles satisfy some other combinatorial criteria. The set of all such polyhedra is infinite, with some partial classifications completed. For example, the fewest sides such a polyhedron may have is four, and the 32 hyperbolic tetrahedra were found by Lannér [6], Vinberg [11], and Thurston [10]. The 825 smallest volume all-right-angled polyhedra have been found by Inoue [4].

For certain computations, it is helpful to know matrix generators in $\mathrm{PSL}_2(\mathbb{C})$ for the orientation-preserving index 2 subgroups of these reflection groups. Such generators were found for some of the tetrahedral groups by Brunner, Lee, and Wielenberg [2], and the second author has shown how to find these for all 32 tetrahedra [5]. These matrix generators have recently been used, for example, by Hoffman [3] to study knot complements which cover tetrahedral orbifolds, and by Şengün [9] to study growth in torsion homology of subgroups of the tetrahedral groups.

In this paper, we perform the same calculations for one class of five-sided polyhedra, which when drawn schematically resemble triangular prisms. The set of all such polyhedra is infinite, and for simplicity we restrict our attention to such polyhedra which have one ideal vertex and five finite vertices. As such, all of our examples are non-compact. We find that the set of all such polyhedra consists of twelve infinite families where one of the dihedral angles can be chosen arbitrarily, and 78 other specific arrangements of dihedral angles.

We note that the list of polyhedra considered in this paper overlaps with those considered by Kaplinskaja [7], who studied finite volume simplicial prisms in $\mathbb{H}^3$, $\mathbb{H}^4$ and $\mathbb{H}^5$ which give rise to discrete reflection groups, and listed their Coxeter graphs. Each of our prisms either appears there, or can be decomposed into two polyhedra which do. As such, our classification is not new, although we do not believe that our polyhedra have previously been listed in this way. The Coxeter graphs of these polyhedra do not immediately allow one to produce isometries which generate the orientation-preserving subgroup, and in this paper we provide a method for this.

The method used is the following. First, we use Andreev's Theorem to set up the combinatorial rules which the dihedral angles must satisfy, and we find all admissible arrangements of angles which satisfy these rules. Then, we reduce the problem of finding hyperbolic planes in the upper half-space $\mathbb{H}^3$ which meet at these prescribed angles to a similar problem, involving finding lines and circles in the plane which meet at the same angles. Finally, we use this geometric data to write down matrix generators for each group.

This paper is organized as follows. After some geometric preliminaries in Section 2, in Section 3 we describe all of the possible arrangements of angles which are possible for our prisms, grouped by the possible angles at the ideal vertex. We also outline a method to locate each prism precisely in $\mathbb{H}^3$. In Section 4 we describe a general method to find each prism and write down corresponding matrices which generate the orientation-preserving subgroup of isometries of $\mathbb{H}^3$, and in Section 5 we summarize the possible angle arrangements in tables.

Acknowledgments. We wish to thank Neil Hoffman for suggesting this project, and for help checking the results, Charles Delman for his guidance and support, and Matthieu Jacquemet for helpful comments on a previous version of this paper.

2 Geometric preliminaries

In this section, we recall some definitions and results about hyperbolic polyhedra. We will work in the upper half-space model for $\mathbb{H}^3$, $\{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$, and we recall that in this model, geodesic lines are vertical lines and semicircles which meet the plane $\{z = 0\}$ perpendicularly, and geodesic planes are vertical planes and hemispheres whose equators lie in the plane $\{z = 0\}$.

We first note that in order for a polyhedron to generate a discrete reflection group, all of its dihedral angles must be integer submultiples of π radians, and the integer must be no less than 2. In this paper, we will label a dihedral angle of π/n by the natural number n.

Definition 2.1. A triangle is *Euclidean* if its angles p, q, and r satisfy the equation

$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} = \pi$$

or

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1.$$

Definition 2.2. A triangle is *spherical* if its angles p, q, and r satisfy the inequality

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

Definition 2.3. A triangle is *hyperbolic* if its angles p, q, and r satisfy the inequality

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

With these definitions in mind, we will appeal to Andreev's Theorem [1] for hyperbolic polyhedra, which specifies exactly what conditions a combinatorial arrangement of dihedral angles must satisfy in order that it give rise to a hyperbolic polyhedron. For the precise statement given below, we refer to Dunbar, Hubbard, and Roeder [8].

Theorem 2.1 (Andreev)**.** *If P is a compact, finite-sided hyperbolic polyhedron with dihedral angle α_i at each edge e_i, then the following conditions hold:*

1. *For each i, $\alpha_i > 0$;*

2. *If three edges* e_i, e_j, e_k *meet at at a vertex, then* $\alpha_i + \alpha_j + \alpha_k > \pi$*;*
3. *If there exists a prismatic* 3*-circuit intersecting* e_i*,* e_j *and* e_k*, then* $\alpha_i + \alpha_j + \alpha_k < \pi$*;*
4. *If there exists a prismatic* 4*-circuit intersecting* e_i*,* e_j*,* e_k *and* e_l*, then* $\alpha_i + \alpha_j + \alpha_k + \alpha_l < 2\pi$*; and*
5. *For a quadrilateral face with edges enumerated successively* e_1*,* e_2*,* e_3 *and* e_4*, and* e_{12}*,* e_{23}*,* e_{34}*, and* e_{41} *are such that* e_{12} *is the third edge meeting at the vertex where* e_1 *and* e_2 *intersect (and similarly for other* e_{ij}*, then*
   (a) $\alpha_1 + \alpha_3 + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi$*; and*
   (b) $\alpha_2 + \alpha_4 + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi$.

The dihedral angle of intersection of two planes in the upper half-space model of $\mathbb{H}^3$ is the same as the angle between the respective tangent planes at any point of intersection. Since these planes are either vertical Euclidean planes or Euclidean spheres with center on the plane $\{z = 0\}$, if two planes intersect, they have a common point on the plane $\{z = 0\}$. In this case, the respective tangent planes are both vertical Euclidean planes, and so the dihedral angle is the angle the tangent planes make in the x-y plane.

With this in mind, we observe that the aim of finding five hyperbolic planes which intersect at prescribed angles may be reduced to finding five lines and circles in the x-y plane which intersect at the same prescribed angles. Since three of our planes intersect at an ideal vertex, we may place this ideal vertex at ∞, and thereby assume that the three planes are vertical Euclidean planes. These will correspond to Euclidean lines in the x-y plane. The remaining two sides will then correspond to circles in the x-y plane; since angles of intersection are preserved by Euclidean similarities, we may take one of the circles to be the unit circle in the x-y plane.

With these assumptions about our setup in place, when finding lines and circles which intersect at given angles, we will appeal frequently to the following results.

Lemma 2.2. *A line that intersects the unit circle at angle* θ *comes* $\cos(\theta)$ *away from the origin at its closest point.*

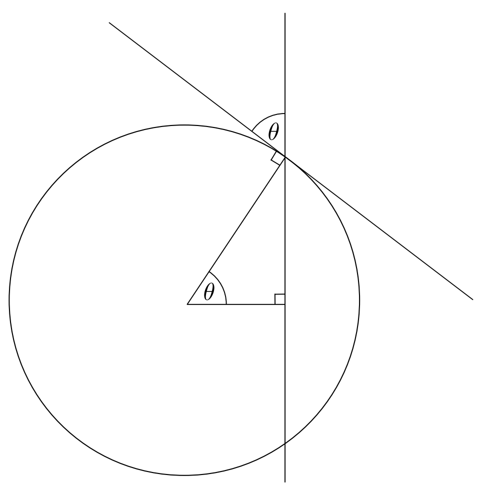

Figure 1: The vertical line is $x = \cos(\theta)$

Proof. After rotating if necessary, we may assume that the line is vertical, and further that it is of the form $x = a$ for $a \geq 0$. The intersection points are then $(a, \sqrt{1-a^2})$ and $(a, -\sqrt{1-a^2})$; we focus on the former. The angle θ is the angle at which the line $x = a$ and the tangent line to the circle at this point intersect (see Figure 1). Since the radius of the unit circle meets the tangent line at a right angle, the triangle with vertices at $(0,0)$, $(a,0)$ and $(a, \sqrt{1-a^2})$ has an angle of $\frac{\pi}{2} - \theta$ at $(a, \sqrt{1-a^2})$. This triangle has a right angle at $(a,0)$, and so must have angle θ at $(0,0)$. Since the hypotenuse of the triangle is a radius of the unit circle, it follows that $a = \cos(\theta)$. ☐

Lemma 2.3. *If a circle with center (x_0, y_0) and radius r meets the line $y = mx + b$ at angle ϕ, and (x_0, y_0) lies on or above the line (that is, $y_0 \geq mx_0 + b$), then x_0, y_0 and r satisfy the equation*

$$y_0 - \frac{r\cos(\phi)}{\sqrt{m^2+1}} = m\left(x_0 + \frac{mr\cos(\phi)}{\sqrt{m^2+1}}\right) + b.$$

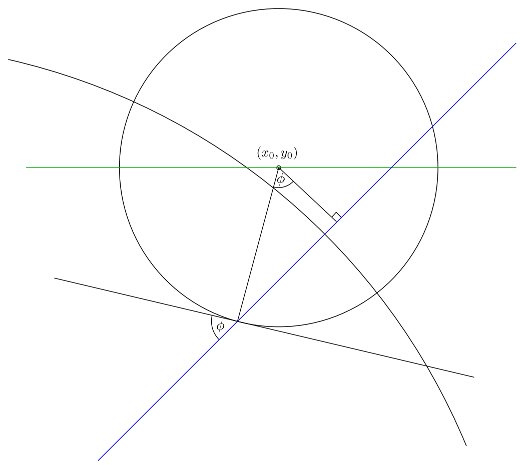

Figure 2: The circle has center (x_0, y_0) and radius r

Proof. The vector $\langle 1, m\rangle$ is parallel to the line, so the vector $\langle m, -1\rangle$ is perpendicular to the line. The unit vector parallel to this is

$$\left\langle \frac{m}{\sqrt{m^2+1}}, \frac{-1}{\sqrt{m^2+1}}. \right\rangle$$

By trigonometry (see Figure 2), we see that the point which is distance $r\cos(\phi)$ away from (x_0, y_0) in the direction of this vector lies on the line. Therefore, plugging the x- and y-co-ordinates of the vector

$$\left\langle x_0 + r\cos(\phi)\frac{m}{\sqrt{m^2+1}}, y_0 - r\cos(\phi)\frac{1}{\sqrt{m^2+1}} \right\rangle$$

into the formula $y = mx + b$ yields the required equation. ☐

Lemma 2.4. *If a circle with center (x, y) and radius r meets the unit circle at angle ϕ, then x, y, and r satisfy the equation*

$$x^2 + y^2 = 1 + r^2 + 2\cos(\phi)r.$$

Proof. This is an application of the Cosine Law to the triangle whose vertices are at $(0,0)$, (x, y), and one of the points where the circles intersect (see Figure 3). This triangle has side lengths, 1, r and $\sqrt{x^2+y^2}$. The angle opposite the latter side is $\pi - \phi$ because the angle between the respective tangent lines at this vertex is ϕ, and the two angles between the tangent lines and their respective radii are both $\pi/2$. The equation follows from the fact that $\cos(\pi - \phi) = -\cos(\phi)$. $\square$

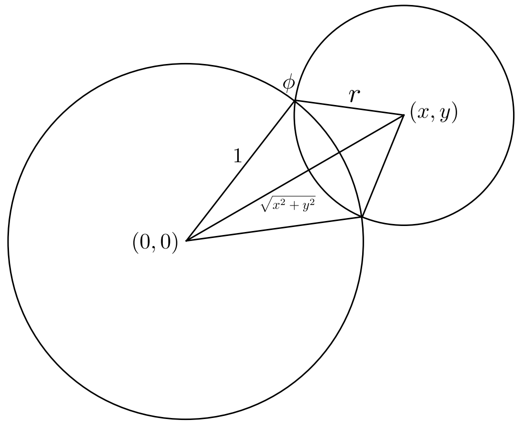

Figure 3: The circles meet at angle ϕ

3 Prisms

In this section, we will describe the dihedral angles of all hyperbolic triangular prisms with one ideal vertex, and a way to construct each prism in the upper half-space model of $\mathbb{H}^3$.

Such a prism is specified by nine positive integers, which we will denote as a_1 through a_9, corresponding to dihedral angles π/a_i. We label the prism as in Figure 4.

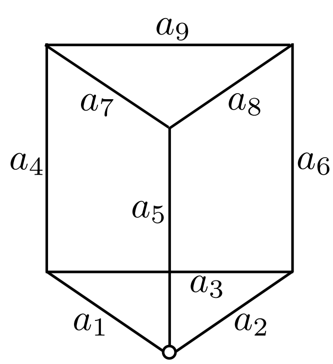

Figure 4: The labels a_1 through a_9

We note that due to the reflectional symmetry of the prism, once we have treated one prism, we have also treated the prism one obtains by exchanging the pairs a_1 and a_2, a_4 and a_6, and a_7 and a_8.

There are restrictions on the combinations of values taken by the labels a_i which correspond to the conditions given in Theorem 2.1. Specifically:

- condition 1 of Theorem 2.1 means that all labels must be positive (i.e. not 0 or ∞; we assume this of the a_i);
- condition 2 states that at the five non-ideal vertices, the three edges incident to the vertex must have the labels of a spherical triangle, and we add here that the three edges incident to the ideal vertex must have the labels of a Euclidean triangle; and
- condition 3 states that labels a_4, a_5 and a_6 must be the labels of a hyperbolic triangle.

We disregard the other two conditions: condition 4 does not apply because the prism has no prismatic 4-circuits; and any labeling of the prism which meets the stated conditions will already meet condition 5. This is because all dihedral angles in question are at most $\pi/2$, at most one of a_4, a_5 and a_6 can be 2 (the others must be larger) from condition 3, and at most one of a_1 and a_2 may be 2 from the ideal vertex condition.

The prisms fall into three categories, and we will handle each separately. These correspond to the angles at the ideal vertex. These labels $[a_1, a_2, a_5]$ must be the labels of a Euclidean triangle, so they are either $[2, 3, 6]$, $[2, 4, 4]$, or $[3, 3, 3]$. For each labeling, we will find three straight lines, and two circles, in the plane, which meet at the prescribed angles.

The three possible arrangements are where p, q, and r of the Euclidean vertex are [2, 3, 6], [2, 4, 4], and [3, 3, 3]. In each case, we will first describe all of the possible labelings, and then find equations for the circles and lines in a few cases. All of the labelings we find are listed in Section 5.

3.1 [2, 3, 6] Cusp

Here a_1, a_2 and a_5 take the values $2, 3$ and 6. We first note that $a_5 \neq 2$. This is because whichever of a_4 or a_6 labels an edge which meets the edge labeled 6 must take the value 2, and then a_4, a_5 and a_6 are not the labels of a hyperbolic triangle.

Let $a_5 = 3$. Then, by symmetry, without loss of generality we suppose $a_1 = 2$ and $a_2 = 6$. Then $a_3 = a_6 = 2$. Since $a_6 = 2$ and $a_5 = 3$, we must have $a_4 \geq 7$, and thus that $a_7 = a_9 = 2$. The remaining label a_8 may take the values $2, 3, 4$ or 5. Each of these 4 cases gives us one infinite family of labelings, indexed by $n \geq 7$ which corresponds to the value of a_4.

Now suppose $a_5 = 6$, and without loss of generality $a_1 = 2$ and $a_2 = 3$. Since $a_5 = 6$, we must have $a_7 = a_8 = 2$. If $a_3 = 4$ then we must have $a_6 = 2$ and $a_4 \leq 3$, in which case a_4, a_5 and a_6 are not the labels of a hyperbolic triangle. A similar argument applies if $a_3 > 4$. If $a_3 = 3$, then we must have $a_6 = 2$, and then a_4 must be 4 or 5. In each of these two cases, a_9 may be 2 or 3.

Finally, if again $a_5 = 6$, $a_1 = 2$, $a_2 = 3$, and $a_7 = a_8 = 2$, it remains to consider $a_3 = 2$. If $a_6 = 2$, then a_4 could be 4 or 5 – in each case a_9 is either 2 or 3 – or $a_4 \geq 6$, in which case $a_9 = 2$. If $a_6 = 3$, then a_4 could be 3, 4 or 5 – if $a_4 = 3$, a_9 is 2, 3, 4 or 5; if a_4 is 4 or 5, a_9 is 2 or 3 – or $a_4 \geq 6$, in which case $a_9 = 2$. If $a_6 = 4$, then a_4 could be 2, 3, 4 or 5 – in each case a_9 is either 2 or 3 – or $a_4 \geq 6$, in which case $a_9 = 2$. If $a_6 = 5$, then a_4 could be 2, 3, 4 or 5 – in each case a_9 is either 2 or 3 – or $a_4 \geq 6$, in which case $a_9 = 2$.

Our first arrangement lets the Euclidean vertex have values of 2, 3, and 6. Let the planes that meet at this vertex be perpendicular to the ground complex plane. Let $a_3 = 2$, $a_4 = 7$, $a_6 = 2$, $a_7 = 2$, $a_8 = 3$, and $a_9 = 2$. This is shown in Figure 5. The values for all the edges were specifically chosen to satisfy definitions 2.1, 2.2, and 2.3. Using these definitions, we can examine different arrangements of the Euclidean vertex with values of 2, 3, and 6.

Making the back quadrilateral face correspond to the unit circle, we choose the red face to correspond to the line $x = 0$, which meets the unit circle at a right angle. By Lemma 2.2, the green face is the line $y = \cos(\pi/7)$, which meets the red face at a right angle, and the unit circle at $\pi/7$. The blue face is the line $y = \sqrt{3}x$, which meets the unit circle at a right angle and the red face at angle $\pi/6$ (see Figure 6).

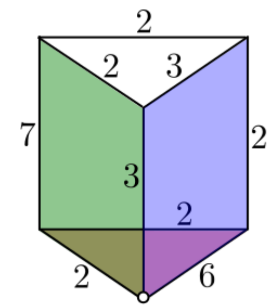

Figure 5: The left quadrilateral face is green; the right is blue; and the lower triangular face is red

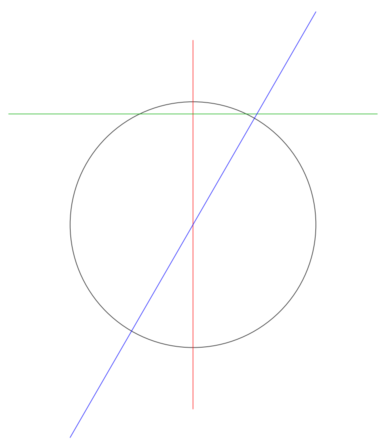

Figure 6: Three lines and the unit circle

It remains to find the last circle, corresponding to the top triangular face of the prism. Suppose this circle has center (x, y) and radius r. This circle intersects the unit circle at an angle of $\pi/2$. Using the Pythagorean Theorem to find an equation, we have that the equation of this intersection is

$$1 + r^2 = x^2 + y^2. \tag{1}$$

Since this circle meets the green line at a right angle, we see that the center must be on the green line, and hence that

$$y = \cos(\pi/7). \tag{2}$$

We need one more equation in x, y and r, and this comes from the fact that the last circle meets the blue line at $\pi/3$. By Lemma 2.3,

$$y - r = \sqrt{3}x \tag{3}$$

(see Figure 7).

These equations, subject to $x > 0$ and $r > 0$, are sufficient to determine x, y and r, and thus to determine the location of the prism precisely. We find that as well as $y = \cos(\pi/7)$, that

$$x = \left(2\sqrt{3}\cos(\pi/7) - \sqrt{6\sin(3\pi/14) - 2}\right)/4 \approx 0.4504$$

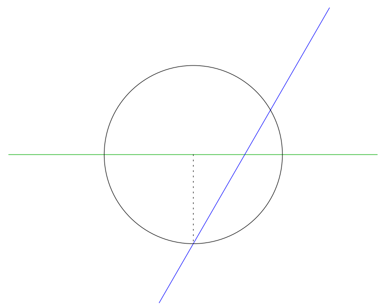

Figure 7: The point $(x, y-r)$ lies on the blue line

and

$$r = \left(\sqrt{18\sin(3\pi/14) - 6} - 2\cos(\pi/7)\right)/4 \approx 0.1209.$$

Now that we have found the equations of the three lines and two circles which intersect at the prescribed angles, the hyperbolic prism we seek is the region inside the triangle defined by the three straight lines, and exterior to the two spheres whose equators are the given circles (see Figure 8).

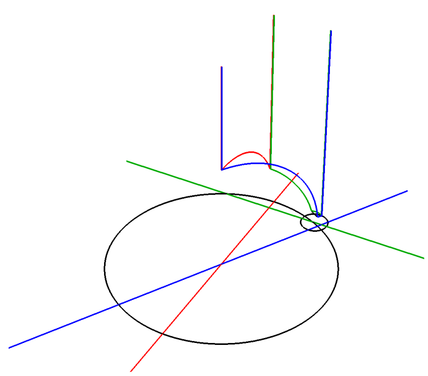

Figure 8: The lines and circles define the hyperbolic planes bounding the prism

Changing the value of a_4 edge from 7 to 8, the equation of the green line changes to $y = \cos(\pi/8)$ by Lemma 2.2. And the same thing happens changing it to equal 9. This result shows that the line gets taller as the value of that side increases. That edge can be left as $a_4 = m > 6$.

Next, we examine another possible arrangement for the $[2, 3, 6]$ prism. Rearranging the values of 2, 3, and 6 at the cusp, we get the following image (see Figure 9).

This image is another specific possibility of the $[2, 3, 6]$ Euclidean vertex. With this arrangement at the cusp, we see that $a_7 = a_8 = 2$, because $a_5 = 6$. By letting a_4 equal 4 and a_6 equal 2, that gives possibilities for letting the a_9 be either a 2 or 3, and a_3 be either a 2 or 3. By plugging in those values, the definitions

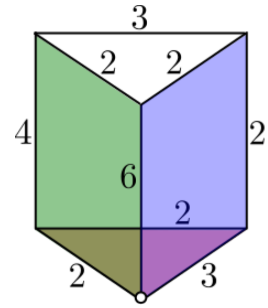

Figure 9: The 2, 3 and 6 at the Euclidean vertex can be rearranged

are satisfied.

We again choose that the back face correspond to the unit circle, and that the red face correspond to the y-axis, $x = 0$. Since a_4 equals a 4, the green face intersects the unit circle at $\pi/4$; since it meets the red face at a right angle, then by Lemma 2.2, the green face corresponds to the line $y = \cos(\pi/4) = \sqrt{2}/2$. The blue face then corresponds to the line $y = (\sqrt{3}/3)x$.

We then find the equation of another circle, corresponding to the top triangular face, and intersecting the unit circle. Since $a_7 = a_8 = 2$, this circle has center on the green and blue lines, which determines the center (x, y) to be at the intersection of these lines. Specifically, $y = \sqrt{2}/2$ and $x = \sqrt{6}/2$. The tangent lines of both circles at their intersections create an angle of $\pi/3$. Since the last circle is intersecting the unit circle, the radius of the unit circle is 1, and the radius of the last circle is r. Using Lemma 2.4, we find that the last equation is $x^2 + y^2 = 1 + r^2 + r$. We find that $r = (\sqrt{5} - 1)/2$.

3.2 [2, 4, 4] Cusp

Next, we examine the arrangement that produces the $[2, 4, 4]$ Euclidean vertex. First, we note that if $a_5 = 2$, then $a_1 = a_2 = 4$. Then a_4 and a_6 are both at most 3, and then a_4, a_5 and a_6 are not the labels of a hyperbolic triangle. Thus we must have $a_5 = 4$, and without loss of generality we will assume $a_1 = 2$ and $a_4 = 4$.

With these assumptions in place, we next note that a_3 must be 2 or 3. Let us first treat the case $a_3 = 3$. In this case, we must have $a_6 = 2$, and then $a_4 = 5$, because it forms a hyperbolic triangle with 2 and 4 and a spherical triangle with 2 and 3. The possible labels for (a_7, a_8, a_9) are then $(2, 2, 2), (2, 2, 3), (2, 3, 2), (2, 3, 3)$ and $(3, 2, 2)$.

Finally, suppose $a_1 = 2$, $a_2 = 4$, $a_3 = 2$, and $a_5 = 4$. Then a_6 is either 2 or 3. If $a_6 = 2$, then $a_4 \geq 5$. If $a_4 = 5$, then as above, the possible labels for (a_7, a_8, a_9) are $(2, 2, 2), (2, 2, 3), (2, 3, 2), (2, 3, 3)$ and $(3, 2, 2)$. If $a_4 \geq 6$, then $a_7 = a_9 = 2$, and a_8 may be 2 or 3. If $a_6 = 3$, then $a_4 \geq 3$. If $a_4 = 3$, then the possible labels for (a_7, a_8, a_9) are $(2, 2, 2), (2, 2, 3), (2, 2, 4), (2, 2, 5), (2, 3, 2)$ and $(3, 2, 2)$. If $a_4 = 4$ or 5, then the possible labels for (a_7, a_8, a_9) are $(2, 2, 2), (2, 2, 3), (2, 3, 2)$ and $(3, 2, 2)$. If $a_4 \geq 6$, then $a_7 = a_9 = 2$, and a_8 may be 2 or 3.

The first possibility of this arrangement we will work with is where a_9 is 3, a_8 is 2, a_7 is 2, a_5 is 4, the a_4 is $m \geq 5$, a_6 is 2. This a_1 is 2, the a_2 is 4, and a_3 is 2 (see Figure 10).

With the bottom vertex being 2, 4, 4, we have a $(90°, 45°, 45°)$ triangle, with blue, red and green sides. The back face of the prism meets green at an angle of $\pi/5$, and the back face meets the blue and red faces at $\pi/2$. The back face corresponds to the unit circle. We choose the red face to correspond to the line $x = 0$. By Lemma 2.2, the green face corresponds to the line $y = \cos(\pi/5)$. The blue line corresponds to the line

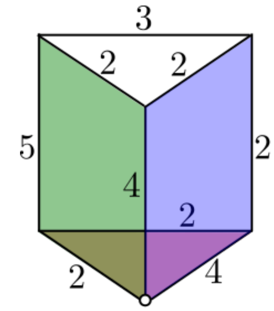

Figure 10: A $[2,4,4]$ cusp

$y = x$.

Finally, the top face meets blue and green at angle $\pi/2$, and the unit circle at angle $\pi/3$. Meeting blue and green at $\pi/2$ means the center of the last circle is on both blue and green lines. Their intersection point is $(x, y) = (\cos(\pi/5), \cos(\pi/5))$. Meeting the unit circle at $\pi/3$ means that, by Lemma 2.4, we have

$$x^2 + y^2 = 1^2 + r^2 - 2\cos\left(\frac{2\pi}{3}\right) r$$

and so

$$x^2 + y^2 = 1 + r^2 + r.$$

Here we find $r = (\sqrt[4]{5} - 1)/2$.

As we saw, with the $[2,4,4]$ cusp the edges a_7, a_8 and a_9 are always labeled 2, 3 or 4, and they cannot all be labeled 3. We now describe what happens to the equations defining x, y and r when these labels change, noting that the red, blue and green faces, as well as the unit circle, are unaffected by these changes.

If we change a_9 to be 2, keeping a_7 and a_8 as 2. This means that the top face intersects the back face at $\pi/2$. The intersection between the last circle and the unit circle creates an angle of $90°$. We still have that $x = y = \cos(\pi/5)$, and the third equation becomes $x^2 + y^2 = 1 + r^2$.

If a_9 is kept as a 2, and we change a_8 to 3, then because the new circle meets the green line at right angles, we still have $y = \cos(\pi/5)$. Also, since $a_9 = 2$, the new circle meets the unit circle at right angles, and so $x^2 + y^2 = 1 + r^2$. By Lemma 2.3, the final equation in this case is

$$y = x + \frac{r\sqrt{2}}{2}.$$

The next arrangement we consider has a_7, a_8, and a_9 equal to 2, the vertical edges as $m \geq 5$, 4, and 2, and the bottom edges as $a_1 = 2$, $a_2 = 4$, and $a_3 = 3$. This change shifts the red face from intersecting the unit circle at $x = 0$ to $x = -1/2$. The green line will be $y = \cos(\pi/a_4)$, and the blue will be $y = x$. Since $a_7 = a_8 = 2$, the center of the second circle is at the intersection of the blue and green lines. Letting a_9 be 2 results in the final equation of the last circle being $1 + r^2 = x^2 + y^2$.

Lastly, keeping a_3 as 3 (so the red line is still $x = -1/2$), we change a_6 to 3. By Lemma 2.2, this shifts the blue line downward from $y = x$ to $y = x - \sqrt{2}/2$. The equations defining x, y and r here are $y = \cos(\pi/a_4)$, $y = x - \sqrt{2}/2$ and $x^2 + y^2 = 1 + r^2$.

3.3 [3, 3, 3] Cusp

The last possible arrangement we will examine is where the Euclidean vertex has values of $[3, 3, 3]$. In studying labelings here, we note that this labeling at the Euclidean vertex is symmetric, and thus we will discard some labelings as being symmetric to other labelings already listed.

We first note that if $a_3 > 2$, then $a_4 = a_6 = 2$ and we do not have a hyperbolic triangle. So $a_3 = 2$. Then by spherical triangles, a_4 and a_6 must both be one of $2, 3, 4$ or 5, but neither can be 2 because of the hyperbolic triangle, and also they cannot both be 3. Because of symmetry considerations, we suppose without loss of generality that $a_4 \leq a_6$.

- If $a_4 = 3$ and $a_6 = 4$, then (a_7, a_8, a_9) can be $(2, 2, 2), (2, 2, 3), (2, 3, 2), (3, 2, 2), (4, 2, 2)$ or $(5, 2, 2)$;
- If $a_4 = 3$ and $a_6 = 5$, then (a_7, a_8, a_9) can be $(2, 2, 2), (2, 2, 3), (2, 3, 2), (3, 2, 2), (4, 2, 2)$ or $(5, 2, 2)$;
- If $a_4 = 4$ and $a_6 = 4$, then (a_7, a_8, a_9) can be $(2, 2, 2), (2, 2, 3)$ or $(2, 3, 2)$ (we discard $(3, 2, 2)$ as it is symmetric to $(2, 3, 2)$);
- If $a_4 = 4$ and $a_6 = 5$, then (a_7, a_8, a_9) can be $(2, 2, 2), (2, 2, 3), (2, 3, 2)$ or $(3, 2, 2)$;
- If $a_4 = 5$ and $a_6 = 5$, then (a_7, a_8, a_9) can be $(2, 2, 2), (2, 2, 3)$ or $(2, 3, 2)$ (we discard $(3, 2, 2)$ as it is symmetric to $(2, 3, 2)$).

The first arrangement we examine is with a_6 equal to 4, $a_4 = 3$, and a_7, a_8, and a_9 equal to 2. As before, we will let the back side of the prism be the unit circle. Since all of the admissible labelings here have $a_3 = 2$, we will fix the red line to be $x = 0$.

Since the blue side meets the back side at $\pi/4$, the blue line intersects the unit circle at $\pi/4$. The green side meets the back side at $\pi/3$, and the red side meets the back side at $\pi/2$. This means that the red side creates a line that passes through the center of the unit circle at angle $90°$. The red, green, and blue lines create an equilateral triangle since their sides meet each other at $\pi/3$.

If we shift the green line so that is passes through the center of the unit circle, we can find the slope of the blue line. Since this line creates an angle of $60°$, in the fourth quadrant, there is a remaining $30°$ in order to make a right angle. Thus the slope of the blue line is

$$\tan\left(\frac{\pi}{6}\right) = \frac{\sin(\pi/6)}{\cos(\pi/6)} = \frac{-1/2}{\sqrt{3}/2} = -\frac{\sqrt{3}}{3}.$$

The slope $-\sqrt{3}/3$ corresponds to the vector $\langle 1, -\sqrt{3}/3\rangle$. To find a vector orthogonal to this vector, we flip one value and negate the other value. We get the vector $\langle 1, \sqrt{3}\rangle$. The unit vector is $\langle 1/2, \sqrt{3}/2\rangle$. Using Lemma 2.2, we multiply the unit vector by $\cos(\pi/3)$ to get the point $(1/4, \sqrt{3}/4)$. Plugging in the slope and this point into the point-slope formula, we get $y = (-\sqrt{3}/3)x + \sqrt{3}/3$.

The slope of the blue line is $\sqrt{3}/3$. The vector corresponding to this slope is $\langle 1, -\sqrt{3}\rangle$ and the unit vector is $\langle 1/2, -\sqrt{3}/2\rangle$. We multiply the unit vector by $\cos(\pi/4)$ to find the point $(\sqrt{2}/4, -\sqrt{6}/4)$. Plugging this point and the slope into the point-slope formula, we get $y = (\sqrt{3}/3)x - \sqrt{6}/3$.

Since a_7, a_8, and a_9 equal 2, that means the last circle intersects the unit circle at right angles. This also means that the center of this last circle lies on the intersection of the green and blue lines. Since the last circle intersects the unit circle at right angles, we use the Pythagorean Theorem to find that the last equation is $1 + r^2 = x^2 + y^2$.

The next possibility is a_6 equal to 5. The only thing that changes is that the blue line has shifted from the center of the unit circle a distance of $\cos(\pi/5)$. We multiply the unit vector by $\cos(\pi/5)$ and plug in the new point into the point-slope formula to get $y = (\sqrt{3}/3)x - (2\sqrt{3}\cos(\pi/5))/3$. The equations for the blue line and the last circle are the same. Here is a list of the equations of the following arrangements where we adjust the values of a_4 and a_6.

Values of a_4, a_5 and a_6: Left to Right	Green Line	Blue Line
3, 3, 4	$y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$	$y = \frac{\sqrt{3}}{3}x - \frac{\sqrt{6}}{3}$
3, 3, 5	$y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$	$y = \frac{\sqrt{3}}{3}x - \frac{2\sqrt{3}}{3}\cos(\frac{\pi}{5})$
4, 3, 4	$y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{6}}{3}$	$y = \frac{\sqrt{3}}{3}x - \frac{\sqrt{6}}{3}$
4, 3, 5	$y = -\frac{\sqrt{3}}{3}x + \frac{\sqrt{6}}{3}$	$y = \frac{\sqrt{3}}{3}x - \frac{2\sqrt{3}}{3}\cos(\frac{\pi}{5})$
5, 3, 5	$y = -\frac{\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}\cos(\frac{\pi}{5})$	$y = \frac{\sqrt{3}}{3}x - \frac{2\sqrt{3}}{3}\cos(\frac{\pi}{5})$

As long as $a_7 = a_8 = a_9 = 2$, the third equation defining the last circle that intersects the unit circle does not change for all of these specific arrangements. Thus, for all of these arrangements, the equation of the last circle is $1 + r^2 = x^2 + y^2$.

Next, we examine what happens when we change a_9 to 3, noting that none of the arrangements for a $[3, 3, 3]$ cusp have a_9 anything other than 2 or 3. The equations for the green and blue lines remain the same as in the list above. The last circle intersects the unit circle at $\pi/3$ instead of $\pi/2$. Since a_7 and a_8 remain a 2, we have that the center of the last circle is on the intersection of the green and blue lines. We use the Law of Cosines (see Lemma 2.4) to find that the equation of the last circle for the arrangements listed above is $x^2 + y^2 = 1 + r^2 + r$.

Next we study the results we get when we change either a_7 to 3 or 4, or a_8 to 3. The equations of the green and blue lines listed above stay the same. The center (x, y) and radius r of the last circle intersecting the unit circle changes. As a_8 changes, the corresponding equation changes according to Lemma 2.3. Setting $\phi = \pi/a_8$, and noting that in all cases the green line has slope $m = \sqrt{3}/3$, with y-intercept b changing, this equation becomes

$$y - \frac{r\cos(\pi/a_8)}{\sqrt{\frac{1}{3}+1}} = \frac{\sqrt{3}}{3}\left(x + \frac{(\sqrt{3}/3)r\cos(\pi/a_8)}{\sqrt{\frac{1}{3}+1}}\right) + b$$

or

$$y = \frac{\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3}r\cos(\pi/a_8) + b.$$

Since a_8 may only takes the values 2 and 3, if it is not 2 then it must be 3, when this equation becomes

$$y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}r + b.$$

By using an analysis like that in the proof of Lemma 2.3, we obtain a similar formula for how the equation corresponding to the green line changes as a_7 changes. If the green line is $y = -(\sqrt{3}/3)x + b$, then our equation is

$$y = -\frac{\sqrt{3}}{3}x - \frac{2\sqrt{3}}{3}r\cos(\pi/a_7) + b.$$

Thus, we have completed all the calculations for all the possible arrangements of a [3, 3, 3] prism.

4 Matrices

In this section, we describe a general method which produces lines and circles which intersect at the prescribed angles, which produces the same results described in the previous section. We then show how to take this geometric data of the lines and circles and use it to produce matrix generators in $\mathrm{PSL}_2(\mathbb{C})$ for the orientation-preserving subgroup of the group generated by reflections in the faces of the corresponding prism. Each group will be generated by four matrices, where each matrix acts by pairing two faces of the polyhedron one obtains by doubling the prism across one face.

4.1 The case $a_3 = 2$

As above, we suppose that the back quadrilateral face lies on the unit sphere, or equivalently that one of the two circles is the unit circle. We further suppose that the red face lies above the imaginary axis, or equivalently that one of the straight lines is $x = 0$; this corresponds to asking that $a_3 = 2$. Lastly, we assume that the polyhedron lies to the right of the imaginary axis as we view it from above; in other words, we assume that it lies in the region of $\mathbb{H}^3$ with $x \geq 0$.

By considering Figure 1 and applying Lemma 2.2, we see that the blue line has equation

$$y = \cot\left(\frac{\pi}{a_2}\right)x - \frac{\cos(\pi/a_6)}{\sin(\pi/a_2)}$$

and, by similar reasoning, the green line has equation

$$y = -\cot\left(\frac{\pi}{a_1}\right)x + \frac{\cos(\pi/a_4)}{\sin(\pi/a_1)}.$$

The final face is on a circle with center (x, y) and radius r, which meets the blue line at angle π/a_8. By Lemma 2.3 we have one equation

$$y - \frac{r\cos(\pi/a_8)}{\sqrt{\cot^2(\pi/a_2)+1}} = \cot(\pi/a_2)\left(x + \frac{r\cot(\pi/a_2)\cos(\pi/a_8)}{\sqrt{\cot^2(\pi/a_2)+1}}\right) - \frac{\cos(\pi/a_6)}{\sin(\pi/a_2)},$$

which simplifies to

$$y - r\cos(\pi/a_8)\sin(\pi/a_2) = \cot(\pi/a_2)\left(x + r\cos(\pi/a_2)\cos(\pi/a_8)\right) - \frac{\cos(\pi/a_6)}{\sin(\pi/a_2)}. \tag{4}$$

By applying Lemma 2.3, with appropriate modifications, to the green line, we also have the equation

$$y + r\sin(\pi/a_1)\cos(\pi/a_7) = -\cot(\pi/a_1)\left(x + r\cos(\pi/a_1)\cos(\pi/a_7)\right) + \frac{\cos(\pi/a_4)}{\sin(\pi/a_1)}. \tag{5}$$

Finally, the two circles intersecting at angle π/a_9 yields, via the Cosine Law and Lemma 2.4, the equation

$$x^2 + y^2 = 1^2 + r^2 - 2(1)(r)\cos\left(\pi - \frac{\pi}{a_9}\right)$$

or

$$x^2 + y^2 = 1^2 + r^2 + 2r\cos\left(\frac{\pi}{a_9}\right). \tag{6}$$

Equations 4, 5 and 6 together define x, y and r and determine the final circle required.

With all of this work in mind, we require seven quantities to write down the matrices we seek. These quantities are $y_1 = \cos(\pi/a_4)/\sin(\pi/a_1)$ and $y_2 = -\cos(\pi/a_6)/\sin(\pi/a_2)$, the y-intercepts of the green and blue line; the angles $\theta_1 = \pi/a_1$ and $\theta_2 = \pi/a_2$ of the ideal triangle at these points; and the center (x, y) and radius r of the second circle. Given these quantities, the matrices are

$$M_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which pairs two sides which both lie on the unit sphere;

$$M_2 = \begin{pmatrix} e^{-i\theta_1} & y_1 i(e^{i\theta_1} - e^{-i\theta_1}) \\ 0 & e^{i\theta_1} \end{pmatrix},$$

which rotates counter-clockwise by angle $2\theta_1$ about $(0, y_1)$;

$$M_3 = \begin{pmatrix} e^{i\theta_2} & y_2 i(e^{-i\theta_2} - e^{i\theta_2}) \\ 0 & e^{-i\theta_2} \end{pmatrix},$$

which rotates clockwise by angle $2\theta_2$ about $(0, y_2)$; and

$$M_4 = \begin{pmatrix} \frac{1}{r}(-x + yi) & \frac{1}{r}(x^2 + y^2) - r \\ \frac{1}{r} & \frac{1}{r}(-x - yi) \end{pmatrix},$$

which sends the second circle to its reflection in the imaginary axis. These matrices satisfy the relations

$$M_2^{a_1} = 1, \quad M_3^{a_2} = 1, \quad M_1^{a_3} = M_1^2 = 1, \quad (M_2^{-1}M_1)^{a_4} = 1, \quad (M_3^{-1}M_2)^{a_5} = 1,$$

$$(M_3^{-1}M_1)^{a_6} = 1, \quad (M_4^{-1}M_2)^{a_7} = 1, \quad (M_4^{-1}M_3)^{a_8} = 1, \quad (M_4^{-1}M_1)^{a_9} = 1.$$

4.2 The case $a_3 = 3$

In the event that $a_3 \neq 2$, then we saw that $a_3 = 3$. In this event, we keep the back face as corresponding to the unit circle, and move the red line to $x = -1/2$ so that it intersects the unit circle at angle $\pi/3$. The equations of the green and blue lines will be the same as the case $a_3 = 2$, and the second circle will be defined by the same three equations 4, 5 and 6. As in the previous case, we define $\theta_1 = \pi/a_1$ and $\theta_2 = \pi/a_2$, and let (x, y) and r be the center and radius of the second circle. In place of y_1 and y_2 we define $z_1 = -1/2 + (\cos(\pi/a_4)/\sin(\pi/a_1) + \cot(\pi/a_1)/2)i$ and $z_2 = -1/2 + (\cos(\pi/a_6)/\sin(\pi/a_2) - \cot(\pi/a_2)/2)i$, the points where the green and blue lines meet the red line $x = -1/2$. Our matrices are then

$$M_1 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

which pairs two sides which lie on the unit sphere and the unit sphere centered at $(-1, 0)$;

$$M_2 = \begin{pmatrix} e^{-i\theta_1} & z_1(e^{i\theta_1} - e^{-i\theta_1}) \\ 0 & e^{i\theta_1} \end{pmatrix},$$

which rotates counter-clockwise by angle $2\theta_1$ about z_1;

$$M_3 = \begin{pmatrix} e^{i\theta_2} & z_2(e^{-i\theta_2} - e^{i\theta_2}) \\ 0 & e^{-i\theta_2} \end{pmatrix},$$

which rotates clockwise by angle $2\theta_2$ about z_2; and

$$M_4 = \begin{pmatrix} \frac{1}{r}(-(x+1) + yi) & \frac{1}{r}(-(x+1) + yi)(-x - yi) - r \\ \frac{1}{r} & \frac{1}{r}(-x - yi) \end{pmatrix},$$

which sends the second circle to its reflection in the line $x = -1/2$. These matrices satisfy the relations

$$M_2^{a_1} = 1, \quad M_3^{a_2} = 1, \quad M_1^{a_3} = M_1^3 = 1, \quad (M_2^{-1}M_1)^{a_4} = 1, \quad (M_3^{-1}M_2)^{a_5} = 1,$$

$$(M_3^{-1}M_1)^{a_6} = 1, \quad (M_4^{-1}M_2)^{a_7} = 1, \quad (M_4^{-1}M_3)^{a_8} = 1, \quad (M_4^{-1}M_1)^{a_9} = 1.$$

5 Results

In this section, we list all of the possible labeling of the prism which were described in Section 3.

5.1 $[2,3,6]$ cusp

5.1.1 $a_3 = 3$

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	3	3	4	6	2	2	2	2
2	3	3	4	6	2	2	2	3
2	3	3	5	6	2	2	2	2
2	3	3	5	6	2	2	2	3

5.1.2 $a_3 = 2$

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	6	2	$n \geq 7$	3	2	2	2	2
2	6	2	$n \geq 7$	3	2	2	3	2
2	6	2	$n \geq 7$	3	2	2	4	2
2	6	2	$n \geq 7$	3	2	2	5	2
2	3	2	4	6	2	2	2	2
2	3	2	4	6	2	2	2	3
2	3	2	5	6	2	2	2	2
2	3	2	5	6	2	2	2	3
2	3	2	$n \geq 6$	6	2	2	2	2
2	3	2	3	6	3	2	2	2
2	3	2	3	6	3	2	2	3
2	3	2	3	6	3	2	2	4
2	3	2	3	6	3	2	2	5
2	3	2	4	6	3	2	2	2
2	3	2	4	6	3	2	2	3
2	3	2	5	6	3	2	2	2
2	3	2	5	6	3	2	2	3
2	3	2	$n \geq 6$	6	3	2	2	2

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	3	2	2	6	4	2	2	2
2	3	2	2	6	4	2	2	3
2	3	2	3	6	4	2	2	2
2	3	2	3	6	4	2	2	3
2	3	2	4	6	4	2	2	2
2	3	2	4	6	4	2	2	3
2	3	2	5	6	4	2	2	2
2	3	2	5	6	4	2	2	3
2	3	2	$n \geq 6$	6	4	2	2	2
2	3	2	2	6	5	2	2	2
2	3	2	2	6	5	2	2	3
2	3	2	3	6	5	2	2	2
2	3	2	3	6	5	2	2	3
2	3	2	4	6	5	2	2	2
2	3	2	4	6	5	2	2	3
2	3	2	5	6	5	2	2	2
2	3	2	5	6	5	2	2	3
2	3	2	$n \geq 6$	6	5	2	2	2

5.2 $[2,4,4]$ cusp

5.2.1 $a_3 = 3$

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	4	3	5	4	2	2	2	2
2	4	3	5	4	2	2	2	3
2	4	3	5	4	2	2	3	2
2	4	3	5	4	2	2	3	3
2	4	3	5	4	2	3	2	2

5.2.2 $a_3 = 2$

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	4	2	5	4	2	2	2	2
2	4	2	5	4	2	2	2	3
2	4	2	5	4	2	2	3	2
2	4	2	5	4	2	2	3	3
2	4	2	5	4	2	3	2	2
2	4	2	$n \geq 6$	4	2	2	2	2
2	4	2	$n \geq 6$	4	2	2	3	2
2	4	2	3	4	3	2	2	2
2	4	2	3	4	3	2	2	3
2	4	2	3	4	3	2	2	4
2	4	2	3	4	3	2	2	5
2	4	2	3	4	3	2	3	2

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	4	2	3	4	3	3	2	2
2	4	2	4	4	3	2	2	2
2	4	2	4	4	3	2	2	3
2	4	2	4	4	3	2	3	2
2	4	2	4	4	3	3	2	2
2	4	2	5	4	3	2	2	2
2	4	2	5	4	3	2	2	3
2	4	2	5	4	3	2	3	2
2	4	2	5	4	3	3	2	2
2	4	2	$n \geq 6$	4	3	2	2	2
2	4	2	$n \geq 6$	4	3	2	3	2

5.3 $[3, 3, 3]$ cusp

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
3	3	2	3	3	4	2	2	2
3	3	2	3	3	4	2	2	3
3	3	2	3	3	4	2	3	2
3	3	2	3	3	4	3	2	2
3	3	2	3	3	4	4	2	2
3	3	2	3	3	4	5	2	2
3	3	2	3	3	5	2	2	2
3	3	2	3	3	5	2	2	3
3	3	2	3	3	5	2	3	2
3	3	2	3	3	5	3	2	2
3	3	2	3	3	5	4	2	2

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
3	3	2	3	3	5	5	2	2
3	3	2	4	3	4	2	2	2
3	3	2	4	3	4	2	2	3
3	3	2	4	3	4	2	3	2
3	3	2	4	3	5	2	2	2
3	3	2	4	3	5	2	2	3
3	3	2	4	3	5	2	3	2
3	3	2	4	3	5	3	2	2
3	3	2	5	3	5	2	2	2
3	3	2	5	3	5	2	2	3
3	3	2	5	3	5	2	3	2

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