THE MOD 2 COHOMOLOGY RINGS OF CONGRUENCE SUBGROUPS IN THE BIANCHI GROUPS

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(WITH AN APPENDIX BY BUI ANH TUAN AND SEBASTIAN SCHÖNNENBECK)

Abstract. We provide new tools for the calculation of the torsion in the cohomology of congruence subgroups in the Bianchi groups : An algorithm for finding particularly useful fundamental domains, and an analysis of the equivariant spectral sequence combined with torsion subcomplex reduction.

1. Introduction

Calegari and Venkatesh have quite recently proven a numerical form of a Jacquet–Langlands correspondence for torsion classes on arithmetic hyperbolic 3-manifolds [6]. This can be seen as a new kind of Langlands programme, for which one has to study torsion in the cohomology of arithmetic groups. A class of arithmetic groups that is of natural interest here, as well as in the classical Langlands programme, are the congruence subgroups in the Bianchi groups. By a Bianchi group, we mean a (P)SL_2 group over the ring of integers in an imaginary quadratic number field. Our aim in this paper is to provide new tools for computing the torsion in the cohomology of the congruence subgroups in the Bianchi groups.

There are already several approaches known for studying congruence subgroups and their cohomology:

(1) Grunewald's method of taking multiples of a fundamental domain for the whole Bianchi group, and deriving presentations for finite index subgroups [8, 9] ;
(2) Utilizing the Eckmann–Shapiro lemma for computing cohomology of congruence subgroups directly from cohomological data of the full Bianchi group [18] ;
(3) Construction of a Voronoï cell complex for the congruence subgroup [4, 19].

Unfortunately, for the following respective reasons none of these approaches were suitable for our investigations.

(1) Grunewald's method works well when the index of the congruence subgroup in the Bianchi group is small. But for larger indices it becomes problematic that one does not know a priori the position of the given congruence subgroup in the list of finite index subgroups through which one has to run. In addition, the list of finite index subgroups grows rapidly with the index.
(2) While the Eckmann–Shapiro lemma allows for a much more efficient computation of the ranks of the cohomology groups, at present this method does not allow one to determine torsion elements of small order in them.
(3) The Voronoï cell complex admits non-trivial and even orientation-reversing actions of its cell stabilizers on their cells, which is an obstruction for studying the geometry of the torsion elements of low order.

Consequently, we introduce an approach which is especially useful for computing the small torsion in the cohomology of our congruence subgroups. What eschews the formulas in [16], is the 2-torsion for the SL_2 subgroups (because the -1 matrix provides constant, omnipresent 2-torsion), and therefore we focus on closing this gap with the present paper.

Our approach is based on a new algorithm, presented in Section 9, which constructs fundamental domains for the congruence subgroups allowing the extraction of torsion subcomplexes. We extend the tools developed in [2] in order to reduce these torsion subcomplexes and to analyze

Date: July 19, 2017.
2000 *Mathematics Subject Classification.* 11F75, Cohomology of arithmetic groups.

the equivariant spectral sequence converging to group cohomology. By analyzing the remaining differentials, we are able to determine almost all of the mod-2 cohomology of the congruence subgroups we consider, and can often get a complete answer. What provides the decisive help here, is a contribution by Norbert Krämer, which allows us to give a complete characterization of all possible 2-torsion subcomplexes which appear in subgroups of Bianchi groups (Section 4). We can then state fairly general formulas for the mod 2 cohomology rings of congruence subgroups in the Bianchi groups (the corollaries in Section 8).

Note that for the above mentioned Voronoï cell complex, since recently we can at least on the machine remedy the non-triviality of the action of cell stabilizers on their cells, using the Rigid Facets Subdivision algorithm [3]. This allows us, in the appendix to this paper, to use Schönnenbeck's computations of the Voronoï cell complex as a check on our example computations, and to illustrate which values the parameters in our formulas can take.

Organization of the paper. In sections 2, 3, 5 and 6, we recall background material from [2]. This includes the definition of the non-central ℓ-torsion subcomplex, whose components provide as efficient way to calculate cohomology rings for the groups we consider. Section 4 states recent results of Krämer, which allow us to categorize all possible non-central ℓ-torsion subcomplexes in congruence subgroups of Bianchi groups for all ℓ. In Section 7 we provide new theorems which calculate the cohomology of all possible reduced 2-torsion subcomplexes. In Section 8 we show that the d_2 differential in our setting is usually 0. Section 9 contains the algorithm we used to construct fundamental domains for the action of the congruence subgroups. Finally, Section 10 provides an example computation.

Acknowledgements. The authors are grateful to the "Groups in Galway" conference series, during a meeting of which that was co-organised by the third author, and attended by the second author and the appendix' second author, the plans for this paper were made. The second author thanks Alan Reid for introducing him to this topic, and for numerous helpful conversations. The third is thankful for being supported by Gabor Wiese's University of Luxembourg grant AMFOR. Special thanks go to Norbert Krämer for helpful suggestions on our manuscript.

2. Preliminaries

In this paper, all cohomology coefficients are the field $\mathbb{F}_2$ with two elements (obviously with the trivial action), unless explicitly mentioned otherwise.

Much of the work in this paper extends the results in [2]. In this section, we provide the major results from that paper without proof, and refer the interested reader to this reference for more details.

2.1. Subcomplex reduction.

Let Γ be any discrete group which acts on a simplicial complex X via a cellular Γ–action so that elements of cell stabilizers fix their cells point-wise. What we have in mind here, is that Γ is a congruence subgroup in a Bianchi group, and acts in the well-known way on hyperbolic 3-space. Let us fix a prime number ℓ. An *ℓ–torsion subcomplex* is the collection of all cells of X whose cell stabilizers contain elements of order a power of ℓ. If, in addition, for every non-trivial finite group $G \subseteq \Gamma$ the fixed point set X^G is acyclic, we have the following special case of Brown's proposition X.(7.2) in [5] :

Proposition 1. *There is an isomorphism between the ℓ–primary parts of the Farrell cohomology of Γ and the Γ–equivariant Farrell cohomology of the ℓ–torsion subcomplex.*

There are instances where the ℓ–torsion complex can be significantly reduced in size yet still yields the appropriate cohomological information. Heuristically, one determines conditions, like isomorphisms on cohomology between stabilizer groups and adjacency conditions, where cells can be merged. The resulting components, the reduced ℓ–torsion subcomplex, can be significantly easier to work with than X itself. Ad hoc use of elements of this approach appears in a number of places in the literature (see [10] or [22], for example.) A systematic reduction procedure was developed by the third author in [14–16]. One of the main results in the latter paper is:

Theorem 2. *There is an isomorphism between the ℓ–primary parts of the Farrell cohomology of Γ and the Γ–equivariant Farrell cohomology of the reduced ℓ–torsion subcomplex.*

In cases where the action of Γ on X has a trivial kernel, one can use the reduced ℓ–torsion subcomplex to determine the Farrell cohomology of Γ [16]. On the other hand, when the kernel contains ℓ–torsion, then the ℓ–torsion subcomplex is all of X, yielding no reduction at all. The following way around this difficulty was developed in [2].

***Definition* 3.** The *non-central ℓ–torsion subcomplex* of a Γ–cell complex X is the union of the cells of X whose cell stabilizers in Γ contain elements of order a power of ℓ, which are not in the center of Γ.

As noted in [2], the *non-central* ℓ–torsion subcomplex for Γ is the same as the ℓ–torsion subcomplex for its central quotient group. In this paper, we use this correspondence to identify the *non-central* ℓ–torsion subcomplex for the action of a congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_{-m})$ on hyperbolic 3–space as the ℓ–torsion subcomplex of the image of the subgroup in $\mathrm{PSL}_2(\mathcal{O}_{-m})$.

To characterize the latter ℓ–torsion subcomplexes, recall that any element of finite order in $\mathrm{PSL}_2(\mathcal{O}_{-m})$ fixing a point inside hyperbolic 3-space $\mathcal{H}$ acts as a rotation of finite order. By Felix Klein's work, we know conversely that any torsion element α is elliptic and hence fixes some geodesic line. So our ℓ–torsion subcomplexes are one-dimensional and consist of rotation axes of elements of finite order.

It is well-known [20] that the finite orders which can occur for elements of $\mathrm{SL}_2(\mathcal{O}_{-m})$ are 1, 2, 3, 4 or 6; so in $\mathrm{PSL}_2(\mathcal{O}_{-m})$ the finite orders are 1, 2 and 3. Hence we have to analyze the 2–torsion subcomplex and the 3–torsion subcomplex. For the Γ-quotient of the 3–torsion subcomplex, theorem 2 of [16] tells us that it consists only of two types of connected components, and the number of these, and hence the mod 3 cohomology of Γ, are explicitly expressed in terms of conjugacy classes of order-3-rotation subgroups and triangle subgroups in Γ. What remains interesting, is the 2–torsion subcomplex, and the explicit determination of the latter will have the emphasis in this paper.

We can now deduce that for $\ell = 2$ or 3, the Γ-quotient of the ℓ–torsion subcomplex is a finite graph (we use either the finiteness of a fundamental domain, or a study of conjugacy classes of finite subgroups as in [12]).

Reducing the 2–torsion subcomplex as mentioned, we can observe the following properties of the reduced non-central 2–torsion subcomplex for a congruence subgroup Γ in $\mathrm{SL}_2(\mathcal{O}_{-m})$:

- All edges of the non-central 2–torsion subcomplex have stabilizer type $\mathbb{Z}/4$ (cyclic with four elements).
- The stabilizer type of the vertices that yield an end-point in the quotient graph is the binary tetrahedral group $\mathbf{Te}$.
- After carrying out the reduction, there are no more vertices with precisely two adjacent vertices.
- The stabilizer type of the vertices that yield a bifurcation in the quotient graph is the quaternion group $\mathbf{Q}_8$ with eight elements.
- Already before the reduction, there are no vertices with more than three adjacent vertices in the quotient graph.

We note that the degree of each vertex in the quotient space is the same as the number of distinct conjugacy classes of $\mathbb{Z}/4$ in the vertex stabilizer. We also observe that all stabilizers which contain a copy of $\mathbf{Q}_8$ are associated to vertices. This observation will be used in Section 5.

2.2. **Spectral sequences.** We use two spectral sequences in this paper. The primary one is the equivariant spectral sequence in cohomology, since it is particularly well-suited to our situation, a cellular action of Γ on the cell complex X. This spectral sequence is developed in detail in [5, Chapter VII]. For our purposes, we note that the E_1 page of the spectral sequence has the form $E_1^{i,j} \cong \prod_{\sigma \in \Gamma \backslash X^{(i)}} \mathrm{H}^j(\Gamma_\sigma)$. where $X^{(i)}$ is a set of Γ-representatives of i–cells in X, and $\Gamma_\sigma \subseteq \Gamma$ is the stabilizer of the cell σ.

We summarize a number of useful properties of the equivariant spectral sequence:

(1) The differential d_1 can be described as the difference of restriction maps in cohomology between cell stabilizers (cohomology analog of [5, VII.8])

$$\prod_{\sigma \in \Gamma\backslash X^{(i)}} \mathrm{H}^j(\Gamma_\sigma) \xrightarrow{d_1^{i,j}} \prod_{\tau \in \Gamma\backslash X^{(i+1)}} \mathrm{H}^j(\Gamma_\tau).$$

(2) There is a product on the spectral sequence, $E_r^{pq} \otimes E_r^{st} \to E_r^{p+s,q+t}$, which is compatible with the standard cup product on $\mathrm{H}^*(\Gamma)$ [5, VII.5]
(3) On the E_2–page, the products in $E_2^{0,*}$, the vertical edge, can be identified with the usual cup product in $\prod_{\sigma\in\Gamma\backslash X^{(0)}} \mathrm{H}^q(\Gamma_\sigma)$. [5, X.4.5.vi].
(4) There are well-defined Steenrod operations, $Sq^k : E_r^{p,q} \to E_r^{p,q+k}$ when $0 \le k \le q$ [21, Theorem 2.15].
(5) When $k \le q-1$, $d_2 Sq^k u = Sq^k d_2 u$, i.e., the Steenrod operations commute with the differential [21, Theorem 2.17].

When calculating restriction maps between finite subgroups in the next section we also use the Lyndon–Hochschild–Serre spectral sequence in cohomology associated to the extension

$$1 \to H \to \Gamma \xrightarrow{\pi} \Gamma/H \to 1, \tag{1}$$

This short exact sequence yields an associated fibration of classifying spaces. The Serre spectral sequence for this fibration has $E_2^{i,j} \cong \mathrm{H}^i(\Gamma/H; \mathrm{H}^j(H;M))$ for untwisted coefficients M and converges to $\mathrm{H}^{i+j}(\Gamma;M)$. For the development of this spectral sequence, we refer readers to either [1, IV.1] or [5, VII.5].

3. Maps induced by finite subgroups in the Bianchi groups

The finite subgroups of the PSL_2 Bianchi groups are well known (see [11], for example), so it is straightforward to determine their preimages in the SL_2 Bianchi groups: the cyclic groups $\mathbb{Z}/2$, $\mathbb{Z}/4$, $\mathbb{Z}/6$, the quaternion group $\mathbf{Q}_8$ of order 8, the dicyclic group $\mathbf{Di}$ of order 12, and the binary tetrahedral group $\mathbf{Te}$ of order 24. The next result lists the cohomology rings for these groups, where polynomial classes are given in square brackets. Since SU_2 acts freely on S^3, the rings are all periodic of period dividing 4.

Proposition 4. *(See [1]) The mod 2 cohomology rings of the finite subgroups of $\mathrm{SL}_2(\mathcal{O}_{-m})$ are:*

$$\begin{aligned}
&\mathrm{H}^*(\mathbb{Z}/4)\cong \mathrm{H}^*(\mathbf{Di}) \cong \mathbb{F}_2[e_2](b_1)\\
&\mathrm{H}^*(\mathbb{Z}/2)\cong \mathrm{H}^*(\mathbb{Z}/6) \cong \mathbb{F}_2[e_1]\\
&\mathrm{H}^*(\mathbf{Q}_8) \cong \mathbb{F}_2[e_4](x_1,y_1)/\langle R\rangle, \ \textit{with } R \textit{ generated by } x_1^2+x_1y_1+y_1^2, x_1^2y_1+x_1y_1^2\\
&\mathrm{H}^*(\mathbf{Te}) \cong \mathbb{F}_2[e_4](b_3)
\end{aligned}$$

The calculation of the cohomology of the reduced 2 torsion subcomplexes from a graph of groups description requires knowledge of restriction maps in cohomology between finite subgroups. The proof of the following Proposition can be found in [2, Proposition 10].

Proposition 5. *The following are the nontrivial restriction maps involving polynomial generators in cohomology for finite subgroups of $\mathrm{SL}_2(\mathcal{O}_{-m})$:*

$$\begin{aligned}
\mathbb{Z}/4:&\quad res^{\mathbb{Z}/4}_{\mathbb{Z}/2}(e_2) = e_1^2\\
\mathbf{Di}:&\quad res^{\mathbf{Di}}_{\mathbb{Z}/6}(e_2) = res^{\mathbf{Di}}_{\mathbb{Z}/2}(e_2) = e_1^2\\
\mathbf{Q}_8:&\quad res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(e_4) = e_2^2,\ res^{\mathbf{Q}_8}_{\mathbb{Z}/2}(e_4) = e_1^4\\
\mathbf{Te}:&\quad res^{\mathbf{Te}}_{\mathbb{Z}/6}(e_4) = res^{\mathbf{Te}}_{\mathbb{Z}/2}(e_4) = e_1^4,\ res^{\mathbf{Te}}_{\mathbb{Z}/4}(e_4) = e_2^2
\end{aligned}$$

In addition, $res^{\mathbb{Z}/6}_{\mathbb{Z}/2}$ and $res^{\mathbf{Di}}_{\mathbb{Z}/4}$ are isomorphisms.

The restriction map between $\mathrm{H}^*(\mathbf{Q}_8)$ and $\mathrm{H}^*(\mathbb{Z}/4)$ is trivial or not depending on the choice of $\mathbb{Z}/4$ subgroup. This is the subject of the next lemma.

Lemma 6. *Given a class in* $\mathrm{H}^1(\mathbf{Q}_8)$*, its image under* $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}$ *is non-trivial for two copies of* $\mathbb{Z}/4 \subseteq \mathbf{Q}_8$ *but trivial on the third.*

Proof. There are three elements of order 2 in $\mathcal{D}_2$, and their central extensions become the three copies of $\mathbb{Z}/4$ in $\mathbf{Q}_8$. Fix x and y, two generators of $\mathcal{D}_2$, viewed multiplicatively, and let x_1 and $y_1 \in H^1(\mathcal{D}_2)$ be their corresponding duals in cohomology. We identify the three subgroups of order 2 in $\mathcal{D}_2$ as $Z_1 = \langle x \rangle$, $Z_2 = \langle y \rangle$, and $Z_3 = \langle xy \rangle$, with corresponding cohomology generators $z_{1,1}$, $z_{1,2}$, and $z_{1,3}$.

Determination of most of the restriction maps $res^{\mathcal{D}_2}_{Z_i}$ is straightforward, but as xy is the product of the distinguished generators, in $res^{\mathcal{D}_2}_{Z_3}$ both x_1 and y_1 are sent to $z_{1,3}$. So

$$\begin{array}{ll} res^{\mathcal{D}_2}_{Z_1}(x_1) = z_{1,1} & res^{\mathcal{D}_2}_{Z_1}(y_1) = 0 \\ res^{\mathcal{D}_2}_{Z_2}(x_1) = 0 & res^{\mathcal{D}_2}_{Z_2}(y_1) = z_{1,2} \\ res^{\mathcal{D}_2}_{Z_3}(x_1) = z_{1,3} & res^{\mathcal{D}_2}_{Z_3}(y_1) = z_{1,3} \end{array}$$

We determine $res^{\mathcal{D}_2}_{Z_i}(x_1 y_1)$ by addition, and the result for $\mathbf{Q}_8$ follows from the Lyndon–Hochschild–Serre spectral sequence associated to

$$1 \to \mathbb{Z}/2 \to \mathbf{Q}_8 \to \mathcal{D}_2 \to 1.$$

□

4. Connected component types of the 2-torsion subcomplex

Corollary 7 (to theorems of Krämer). *Let* $\Gamma_0(\eta)$ *be a congruence subgroup in* $\mathrm{SL}_2(\mathcal{O})$*, where* $\mathcal{O}$ *is the ring of integers in the imaginary quadratic field* $\mathbb{Q}(\sqrt{-m})$ *(m square-free) with discriminant* $\Delta \neq -4$*, and subject to the congruence condition that the lower left entry is contained in the ideal* $\eta \subsetneq \mathcal{O}$*. Let t be the number of distinct prime divisors of* Δ*. We consider the action of* $\Gamma_0(\eta)$ *on the associated symmetric space, hyperbolic 3-space. Then the reduced non-central 2-torsion subcomplex consists exclusively of connected components of types , , and , with multiplicities as follows.*

- For every connected component of type , there must be two conjugacy classes of binary tetrahedral subgroups in $\Gamma_0(\eta)$. Such subgroups exist precisely when $\eta = \langle 2 \rangle$, $m \equiv 3 \bmod 8$ and for all prime divisors p of Δ, $p \equiv 1$ or $3 \pmod 8$. There are then precisely 2^{t-1} connected components of this type.
- & The existence of either type and requires two conjugacy classes of maximal quaternion 8-groups per connected component. The congruence subgroup $\Gamma_0(\eta)$ contains maximal quaternion 8-subgroups if and only if $m \not\equiv 3 \bmod 4$ and in addition,
  - either $\eta = \langle 2 \rangle$ and $p \equiv 1 \bmod 4$ for all odd prime divisors p of Δ,
  - or $\eta^2 = \langle 2 \rangle$ and for all divisors $n \in \mathbb{N}$ of Δ, we have $n \not\equiv 7 \bmod 8$.
- A sufficient condition for all maximal quaternion 8-groups to sit on -components is that $\eta^2 = \langle 2 \rangle$, $m \equiv 2 \bmod 4$ and there exist $x, y \in \mathbb{Z}$ with $x^2 - my^2 = 2$. When this condition holds, there are 2^{t-1} connected components of type .
- Conversely, sufficient conditions for all maximal quaternion 8-groups to sit on -components are that $\eta = \langle 2 \rangle$; or that $m \not\equiv 2 \bmod 4$; or that $x^2 - my^2 \neq 2$ for all $x, y \in \mathbb{Z}$. The number of connected components of type is then
  - 2^{t-1}, if $\eta = \langle 2 \rangle$.
  - 2^{t-1}, if $\eta^2 = \langle 2 \rangle$ and $p \equiv 1 \bmod 8$ for all odd prime divisors p of Δ.
  - 2^{t-2}, if $\eta^2 = \langle 2 \rangle$ and $p \equiv \pm 3 \bmod 8$ for some prime divisor p of Δ.

  Note that the existence conditions directly imply that $p \not\equiv 7 \bmod 8$.

The remaining conjugacy classes of cyclic groups of order 4 (i.e., those not involved in the components mentioned above) constitute components.

Proof. The theorems that we quote are results in Krämer's preprint [12]. We refer the reader there for more details. We first provide a summary of the overall argument. Satz 9.7 implies that every stabilizer group of binary tetrahedral type occurs only as an endpoint of a component of type . For maximal quaternion-8 stabilizer subgroups, the conditions for the existence of components of types and are complementary, so these are the only components that can occur in this case. The remaining connected components admit exclusively stabilizers cyclic of order four, so they are of type .

For the individual component types, we observe

- Satz 9.4.(i) states the conditions given in this Corollary for the existence of binary tetrahedral groups. Satz 9.6.(i) specifies the number of conjugacy classes of binary tetrahedral type as 2^t. There are two such conjugacy classes needed for each component.
- & The proof of conditions for existence are given in Satz 9.4.(iii).
- The proof of the sufficient condition for is given with Satz 9.9.(i). This condition yields, as is stated in Satz 9.9.(i), that for all odd prime divisors p of Δ, we have $p \equiv 1 \pmod 8$. Therefore by Satz 9.6.(iii), there are 2^t conjugacy classes of maximal quaternion 8-groups.
- The proof of the sufficient conditions for is given in Satz 9.9.(ii). The proof of the number of conjugacy classes is given in Satz 9.6.(iii).

□

Examples. We have explicitly computed the type and number of components for a number of fundamental domains computed with the algorithm in Section 9. The following results for the non-central s-torsion subcomplex quotient, ${}_{\Gamma}\backslash X_s$ for $s \in \{2, 3\}$, are in accordance with Corollary 7.

	Γ	${}_{\Gamma}\backslash X_3$	${}_{\Gamma}\backslash X_2$
(1)	$\Gamma_0(1+\sqrt{-2}) \subset \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$		empty
(2)	$\Gamma_0(2) \subset \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$	empty	
(3)	$\Gamma_0(5) \subset \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$		
(4)	$\Gamma_0(\sqrt{-2}) \subset \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$	empty	
(5)	$\Gamma_0(3+2\sqrt{-2}) \subset \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$	empty	
(6)	$\Gamma_0(2) \subset \mathrm{SL}_2(\mathbb{Z}[\frac{-1+\sqrt{-11}}{2}])$		
(7)	$\Gamma_0(\frac{-1+\sqrt{-11}}{2}) \subset \mathrm{SL}_2(\mathbb{Z}[\frac{-1+\sqrt{-11}}{2}])$		empty

Details for Examples (3) and (4) above are given in Section 10.

5. Calculation of the second page of the spectral sequence

In this section we summarize results from [2] concerning the generic structure of the equivariant spectral sequence in our setting. For more details, the reader should consult [2].

We start with a number of definitions. There is an action via linear fractional transformations of the congruence subgroup Γ on hyperbolic 3-space, $\mathbb{H}^3$. Let X be the 2-dimensional cellular retraction of $\mathbb{H}^3$ from which we build the Ford domain for the action of Γ (see [17]). We denote by X_s the non-central 2-torsion subcomplex of X, and by X'_s the subcomplex of X consisting of vertices whose stabilizer is either $\mathbf{Q}_8$ or $\mathbf{Te}$.

We define c to be the rank of the cokernel of $\mathrm{H}^1({}_{\Gamma}\backslash X; \mathbb{F}_2) \to \mathrm{H}^1({}_{\Gamma}\backslash X_s; \mathbb{F}_2)$ induced by the inclusion $X_s \subset X$. That is, the case $c > 0$ corresponds to the situation where loops in the non-central 2-torsion subcomplex are identified in the quotient space. We also define v to be the number of conjugacy classes of subgroups isomorphic to $\mathbf{Q}_8$ in Γ; these occur in vertex stabilizers isomorphic to $\mathbf{Q}_8$ or $\mathbf{Te}$. There is a geometric meaning for v, related to the 2-torsion subcomplex components found in X as listed in Section 4. Specifically, it is shown in [2] that v counts the number of vertices in the non- 2-torsion subcomplex components of X. Finally, we define $\mathrm{sign}(v)$ to equal 0 when $v = 0$ and 1 otherwise. The following theorem is one of the main results in [2], we only have to add the trivial case where X_s is empty; the contrary was assumed in proposition 19 of the quoted paper because only full Bianchi groups were considered, but it is straightforward to work out the missing case.

Theorem 8. [2, Corollary 21] *The E_2 page of the equivariant spectral sequence with $\mathbb{F}_2$–coefficients associated to the action of Γ on X is concentrated in the columns $n \in \{0, 1, 2\}$. If X_s is empty, then $E_2^{n,q} \cong \mathrm{H}^n({}_\Gamma\backslash X; \mathbb{F}_2)$ for all $n, q \geq 0$; otherwise this E_2 page has the following form.*

$q = 4k+3$	$E_2^{0,3}(X_s)$	$E_2^{1,3}(X_s) \oplus (\mathbb{F}_2)^{a_1}$	$(\mathbb{F}_2)^{a_2}$
$q = 4k+2$	$\mathrm{H}^2_\Gamma(X'_s) \oplus (\mathbb{F}_2)^{1-\mathrm{sign}(v)}$	$(\mathbb{F}_2)^{a_3}$	$\mathrm{H}^2({}_\Gamma\backslash X)$
$q = 4k+1$	$E_2^{0,1}(X_s)$	$E_2^{1,1}(X_s) \oplus (\mathbb{F}_2)^{a_1}$	$(\mathbb{F}_2)^{a_2}$
$q = 4k$	$\mathbb{F}_2$	$\mathrm{H}^1({}_\Gamma\backslash X)$	$\mathrm{H}^2({}_\Gamma\backslash X)$
$k \in \mathbb{N} \cup \{0\}$	$n = 0$	$n = 1$	$n = 2$

where

$$\begin{aligned} a_1 &= \chi({}_\Gamma\backslash X_s) - 1 + \beta^1({}_\Gamma\backslash X) + c \\ a_2 &= \beta^2({}_\Gamma\backslash X) + c \\ a_3 &= \beta^1({}_\Gamma\backslash X) + v - \mathrm{sign}(v) \end{aligned}$$

and where $\beta^q = \dim_{\mathbb{F}_2} \mathrm{H}^q({}_\Gamma\backslash X; \mathbb{F}_2)$ for $q = 1, 2$; and $\chi({}_\Gamma\backslash X_s)$ is the usual Euler characteristic of the orbit space of the 2-torsion subcomplex ${}_\Gamma\backslash X_s$.

Let us note here that $\dim_{\mathbb{F}_2} \mathrm{H}^2_\Gamma(X'_s)$ is twice the number of orbits of vertices of stabilizer type $\mathbf{Q}_8$ (cf. [2, note 30]).

6. Calculation of the spectral sequence on the subcomplex

In Theorem 8, we have expressed the E_2 page of the equivariant spectral sequence converging to $\mathrm{H}^*(\mathrm{SL}_2(\mathcal{O}_{-m}))$ in terms of invariants of the quotient space and the E_2 page associated to the non-central 2-torsion subcomplex X_s. In this section, we will investigate that latter E_2 page, $E_2^{p,q}(X_s)$. Recall that the only 2-torsion elements which stabilize cells outside of the non-central 2–torsion subcomplex are in the $\mathbb{Z}/2$ center. This allows us to relate back to and use results about the 2–torsion subcomplex for $\mathrm{PSL}_2(\mathcal{O}_{-m})$. What we have to do in order to establish this relation is to show that $E_2^{p,q}(X_s)$ splits as a direct sum indexed by the connected components of X_s; then further that the reduction of subcomplex components is in complete agreement with the analogous reduction for $\mathrm{PSL}_2(\mathcal{O}_{-m})$. We do this in the following lemma.

Lemma 9. *The terms $E_2^{p,q}(X_s)$ split into direct summands each with support on one connected component of the quotient of the* reduced *non-central 2–torsion subcomplex (again denoted X_s).*

7. Cohomology of 2-torsion subcomplexes

In this section we will calculate the mod 2 cohomology of a generic component of the non-central reduced 2-torsion subcomplex via a graph of groups description and the equivariant spectral sequence. We use Proposition 5 and Lemma 6 to determine the d_1 differential. By the periodicity of cohomology, it is sufficient to restrict ourselves to dimensions $q \leq 4$. The restriction map between $\mathrm{H}^1(\mathbf{Q}_8)$ and $\mathrm{H}^1(\mathbb{Z}/4)$ requires the most attention, since each $\mathbf{Q}_8$ vertex stabilizer contains three non-conjugate copies of $\mathbb{Z}/4$, a fact reflected in the three edges incident to that vertex. There are two cases, according as whether two of the incident edges form a loop or not. We call an edge which forms a loop a *looped edge*, or simply a *loop*.

Lemma 10. *Let $\mathbf{Q}_8$ be a vertex stabilizer with no looped edges. Then under d_1, any class in $\mathrm{H}^1(\mathbf{Q}_8)$ restricts isomorphically to exactly two copies of $\mathrm{H}^1(\mathbb{Z}/4)$ and is trivial on the third.*

Proof. This follows directly from Lemma 6. We note that the three classes in $\mathrm{H}^1(\mathbf{Q}_8)$ are detected on different pairs of $\mathbb{Z}/4$ subgroups. □

Lemma 11. *Let $\mathbf{Q}_8$ be a vertex stabilizer with adjacent to a looped edge, and let $b_{1,1}$ and $b_{1,2}$ be the classes in $\mathrm{H}^1(\mathbb{Z}/4)$ associated to the looped edge and unlooped edge respectively. Then under d_1, any class in $\mathrm{H}^1(\mathbf{Q}_8)$ is either detected by the cohomology of both edge stabilizers, or neither.*

Proof. Let y be the group element of Γ associated with the unlooped edge, and x and xy be the group elements of Γ associated with one side of the loop. We again use Lemma 6. However, we note that the subgroup Z_3 is now identified with Z_1, so in cohomology $b_{1,1} = b_{1,3}$. For the unlooped edge stabilizer, associated to Z_2, we have

$$res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(x_1) = 0 \quad res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(y_1) = b_{1,2}$$

The restriction to the looped edge stabilizer is induced by the difference of restriction maps to Z_1 and Z_3 in Lemma 6. Therefore, this restriction map is given by

$$res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(x_1) = b_{1,1} - b_{1,1} = 0 \quad res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(y_1) = b_{1,1}$$

We detemine the restriction on $x_1 + y_1$ by additivity, and the result follows. □

It is usually clear by Proposition 5 which classes are in the kernel of d_1 and which are in its image. However, Lemmas 10 and 11 show that the situation is more subtle in dimension 1. We introduce a graphical idea which will aid us in determining $\ker d_1$ in this case. First, note that the only classes in $E_2^{0,1}$ come from copies of $\mathbf{Q}_8$ vertex stabilizers. Therefore, any class in $E_2^{0,1}$ can be written as $\sigma \in \oplus_S \mathrm{H}^1(\mathbf{Q}_8)$ for some index set S. By Property 1 in Subsection 2.2, the map d_1 is the difference of restriction maps for vertex stabilizer groups to edge stabilizer groups. We make the following observation about non-looped edges e with endpoints v_0 and v_1: for a class to be in $\ker d_1$, a necessary condition (mod 2) is that the restriction maps to the edge stabilizers must both vanish, or both must be non-trivial. That is, for non-looped edges,

$$res^{\Gamma_{v_1}}_{\Gamma_e} = res^{\Gamma_{v_0}}_{\Gamma_e}. \tag{2}$$

We understand the restriction maps from $\mathrm{H}^1(\mathbf{Q}_8)$. When there are no looped edges, a class in $\mathrm{H}^1(\mathbf{Q}_8)$ is detected on exactly two edges by Lemma 10; and when there is a looped edge, by Lemma 11, a class in $\mathrm{H}^1(\mathbf{Q}_8)$ is either detected on both the looped and unlooped edge, or it is sent to 0.

We remark that it is possible to order basis elements in such a way that d_1 is described by a block matrix where each block is associated to a single connected component C. Therefore, we can analyze $d_1^{0,1}$ one component at a time. We have our first result.

Lemma 12. *For each looped edge in a non-central 2-torsion subcomplex quotient C, there is a class which is in $\ker d_1^{0,1}$.*

Proof. The class in $\ker d_1^{0,1}$ is x_1 from Lemma 11. □

Given a cohomology class $\sigma \in \oplus_S \mathrm{H}^1(\mathbf{Q}_8)$ from a non-central 2-torsion subcomplex quotient component C, the *support* of σ is a subgraph of C built as follows: For each copy of $G \cong \mathbf{Q}_8$ which contributes to σ, add to the subgraph its associated vertex, and the incident edges which detect $\sigma|_G$ as given by Lemmas 10 and 11.

Lemma 13. *If $\sigma \in \oplus_S \mathrm{H}^1(\mathbf{Q}_8)$ has support which includes an edge incident to a leaf or a loop in C, then $\sigma \notin \ker d_1^{0,1}$.*

Proof. We note that the only reduced non-central 2-torsion subcomplex quotient components which have $\mathbb{Z}/4$ vertex stabilizers are loops, so they do not fall under the hypothesis of this lemma. Like all edge stabilizers in C, the stabilizer of e, $Stab(e)$, is isomorphic to $\mathbb{Z}/4$. There are two cases to consider, depending on whether the edge, e, is adjacent to a leaf or a looped edge. In both cases we will focus on a $\mathbf{Q}_8$ vertex stabilizer on the other side of e. We denote this distinguished copy of $\mathbf{Q}_8$ by Q.

In the first case, e is adjacent to a leaf, which, as a vertex of degree 1, has a **Te** stabilizer. Since $\mathrm{H}^1(\mathbf{Te}) = 0$, there can be no non-trivial restriction maps in degree 1 from the cohomology of the vertex stabilizer to $\mathrm{H}^1(Stab(e))$. Therefore, if σ is supported on e, this happens because $res^{Q}_{Stab(e)}$ is nontrivial. In this case, Equation 2 is not satisfied, and $\sigma \notin \ker d_1^{0,1}$.

For the other case, e is adjacent to a looped edge. By Lemma 11, the restriction map in cohomology from the vertex group adjacent to the looped edge either maps non-trivially to both edge groups or to neither. If both, then there can be no second restriction map to the cohomology of the looped edge to make $d_1^{0,1}$ vanish. If neither, then the support arises because $res^{Q}_{Stab(e)} \neq 0$. By Equation 2, such a class cannot be in $\ker d_1^{0,1}$. □

Corollary 14. *$\sigma \in \ker d_1^{0,1}$ is supported on a tree, then $\sigma \equiv 0$.*

We now to proceed to the general case. Given a reduced non-central 2-torsion subcomplex quotient component C, form a connected subcomponent C' by removing leaf vertices, vertices adjacent to looped edges, and all edges adjacent to these vertices. The result of Lemma 13 implies that if classes in $\ker d_1^{0,1}$ do not originate from looped edges as in Lemma 12, then they must be supported on C'. We note that all vertex stabilizers in C' are isomorphic to $\mathbf{Q}_8$. We also highlight one degenerate case, where there are two vertices connected by an edge, each with a looped edge. Here, we apply Lemma 12 to see that for the block of $d_1^{0,1}$ supported on this component, $\dim_{\mathbb{F}_2} \ker d_1^{0,1} = 2$.

Lemma 15. *The block of $d_1^{0,1}$ supported on C' satisfies $\dim_{\mathbb{F}_2}(\ker d_1^{0,1}|_{C'}) = \beta_1(C')$, where $\beta_1(C')$ is the first Betti number of C'.*

Proof. We can associate to any cycle in C' a class σ in $\ker d_1^{0,1}$. The proof is by construction. For each vertex in cycle with stabilizer Q, choose the unique class in $\mathrm{H}^1(Q)$ which restricts to the two edges of the chosen cycle. Use these to form $\sigma \in \oplus_S \mathrm{H}^1(\mathbf{Q}_8)$. Since Equation 2 is satisfied, $\sigma \in \ker d_1^{0,1}$.

Next, pick a maximal rooted spanning tree T of C'. Enumerate the remaining edges of $C' \setminus T$ by $e_1, e_2, \ldots, e_k$ and give these edges an arbitrary orientation. We associate to each edge e_j a cycle c_j as follows: Travel the unique path from the tree root to the tail of e_j, traverse the edge, then follow the unique path back through the tree to the root. We remove from the cycle edges which are traversed twice during this process.

We construct a cohomology class in $\ker d_1^{0,1}$ associated to this cycle as above, and denote it by s_j. The collection $\{s_1, s_2, \ldots, s_k\}$ form a linearly independent set, since each class s_j is supported on an edge, e_j, that is part of no other class. We claim that the $\{s_j\}$ span $\ker d_1^{0,1}$.

Take an arbitrary element $\sigma \in \ker d_1^{0,1}$ supported on C', and let E' be the set of edges in σ's support. We consider the cohomology class

$$\sigma' = \sigma - \sum_{e_j \in E'} s_j.$$

We claim that σ' is supported on the tree T. In particular, by Equation 2, we know that each edge $e_j \in E'$ in the support of σ detects the cohomology of both its vertex stabilizers. The same edge in $\sigma - s_j$, however, does not. We can apply Corollary 14 to conclude that $\sigma' \equiv 0$. Therefore

$$\sigma = \sum_{e_j \in E'} s_j.$$

Finally, in a graph, the number of distinguished edges $\{e_j\}$ is the same as the first Betti number of the graph. □

We combine Lemmas 12 and 15 for our main result.

Theorem 16. *Given C, a reduced non-central 2-torsion subcomplex quotient component, the block of $d_1^{0,1}$ supported on C satisfies $\dim_{\mathbb{F}_2}(\ker d_1^{0,1}|_C) = \beta_1(C)$, where $\beta_1(C)$ is the first Betti number of C.*

Proof. From Lemma 12, each looped edge results in exactly one class in $\ker d_1^{0,1}$. This set is disjoint from the results from Lemma 15. □

In our non-central reduced 2–torsion subcomplexes, edge stabilizers are always of type $\mathbb{Z}/4$; and vertex stabilizers are of one of the types

- **Te**, with one edge adjacent to the vertex in the quotient space, or
- $\mathbb{Z}/4$, with one edge adjacent at both of its ends, yielding a connected component , or
- $\mathbf{Q}_8$, with three edges (counted once or twice) adjacent to the vertex in the quotient space.

Corollary 7 tells us that the only possible types of connected components in our quotients of reduced non-central 2–torsion subcomplexes are , , and . In the following theorem, we treat them in a unified way, which would allow also for other shapes of connected components that we could construct as graphs, using the branching that takes place at vertices of stabilizer type $\mathbf{Q}_8$. For this purpose, we exclude connected components of type ○•, with the consequence that the only types of vertex stabilizers on our connected component are $\mathbf{Q}_8$ and $\mathbf{Te}$.

Theorem 17. *Let C be a connected component, not of type ○•, in some reduced 2 torsion subcomplex where m vertices have $\mathbf{Q}_8$ stabilizers, n vertices have $\mathbf{Te}$ stabilizers, and with the homotopy type of an l-looped rose. Then*

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(C) = \begin{cases} 1, & q = 0; \\ m+n+l, & q = 4k, \ k \geq 1; \\ 2l, & q = 4k+1; \\ n+m+2l-1 & q = 4k+2; \\ 2m+2n+l-1, & q = 4k+3. \end{cases}$$

Proof. The component C contains $m+n+l-1$ edges, all of which have $\mathbb{Z}/4$ stabilizer. The E_1 and E_2 pages of the equivariant spectral sequence are as follows.

$$\begin{array}{c|lcl} & & E_1 \text{ page} & \\ 4 & \mathbb{F}_2^{m+n} & \longrightarrow & \mathbb{F}_2^{m+n+l-1} \\ 3 & \mathbb{F}_2^{m+n} & \longrightarrow & \mathbb{F}_2^{m+n+l-1} \\ 2 & \mathbb{F}_2^{2m} & \longrightarrow & \mathbb{F}_2^{m+n+l-1} \\ 1 & \mathbb{F}_2^{2m} & \longrightarrow & \mathbb{F}_2^{m+n+l-1} \\ 0 & \mathbb{F}_2^{m+n} & \longrightarrow & \mathbb{F}_2^{m+n+l-1} \\ \hline & 0 & & 1 \end{array} \quad \xrightarrow{\mathrm{H}_*(d_1)} \quad \begin{array}{c|ll} & E_2 \text{ page} & \\ 4 & \mathbb{F}_2 & \mathbb{F}_2^{l} \\ 3 & \mathbb{F}_2^{m+n} & \mathbb{F}_2^{m+n+l-1} \\ 2 & \mathbb{F}_2^{2m} & \mathbb{F}_2^{m+n+l-1} \\ 1 & \mathbb{F}_2^{l} & \mathbb{F}_2^{n-m+2l-1} \\ 0 & \mathbb{F}_2 & \mathbb{F}_2^{l} \\ \hline & 0 & 1 \end{array}$$

To get to the E_2 page we need to determine the d_1 differential. At the level of stabilizers, in dimensions $\equiv 0 \bmod (4)$, the inclusions are based on the shape of C, and C is homotopic to an l-leafed rose. In dimensions $\equiv 1 \bmod (4)$, Theorem 16 implies that $\dim \ker d_1^{0,1} = l$. And from Proposition 5, we know that d_1 vanishes in dimensions $\equiv 2, 3 \bmod (4)$. □

For the excluded case, we recall a lemma that has already been established.

Lemma 18 ([2, lemma 26])**.** *Let C be a connected component of type ○•. Then* $\dim_{\mathbb{F}_2} \mathrm{H}^q(C) = \begin{cases} 1, & q = 0; \\ 2, & q \geq 1 \end{cases}$ *and $E_2^{p,q}|_C \cong \mathbb{F}_2$ for all $q \geq 0$, $p \in \{0, 1\}$.*

8. Determination of the d_2 differential

In order to complete our cohomology calcuations, we need to understand the d_2 differential in the equivariant spectral sequence. In general this can be quite difficult, but in our situation the existence of Steenrod operations in the spectral sequence help out a great deal. We follow the approach laid out in [2], quoting results as needed. We treat first the degenerate case where the non-central 2-torsion subcomplex X_s is empty, so we can afterwards assume that it is non-empty.

Proposition 19. *Let the non-central 2-torsion subcomplex X_s be empty. Then*

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\Gamma; \mathbb{F}_2) = \begin{cases} \beta^1 + 1, & q = 1, \\ \beta^2 + \beta^1 + 1, & q \geq 2, \end{cases}$$

where $\beta^q = \dim_{\mathbb{F}_2} \mathrm{H}^q({}_{\Gamma}\backslash X; \mathbb{F}_2)$.

Proof. In view of Theorem 8, we only have to show that the d_2 differential vanishes completely. For this purpose, we consider the homological equivariant spectral sequence with integer coefficients. That sequence is, modulo 3-torsion and apart from the zeroth row, concentrated in the odd rows, with (p,q)th entry $(\mathbb{Z}/2)^{\beta^p}$. Therefore its d_2 differential is zero modulo 3-torsion in rows above $q = 1$. Comparing via the Universal Coefficient theorem, we see that the d_2 differential in rows above $q = 1$ must vanish also for the cohomological equivariant spectral sequence with mod 2 coefficients. By the periodicity, we obtain that the latter differential vanishes also in low degrees. □

Recall from Section 5 the definition of c as the rank of the cokernel of

$$\mathrm{H}^1({}_{\Gamma}\backslash X;\, \mathbb{F}_2) \to \mathrm{H}^1({}_{\Gamma}\backslash X_s;\, \mathbb{F}_2)$$

induced by the inclusion $X_s \subset X$.

Lemma 20. [2, Lemma 31] *When $c = 0$, the d_2 differential vanishes on $E_2^{0,4q+2}$.*

Proof. Although this proof is contained in [2], the hypothesis that $c = 0$ is missing there. We sketch the argument, indicating the need for $c = 0$.

We note that Sq^1 vanishes on $\mathrm{H}^2(G)$ for all finite groups G which appear as vertex stabilizers. In the equivariant spectral sequence, a non-zero target of the d_2 differential is an odd-dimensional class in $\oplus\,\mathrm{H}^*(\mathbb{Z}/2)$ in the second column, where the sum is over 2-cells in ${}_{\Gamma}\backslash X$. Looking back at the derivation of Theorem 8, this class has non-trivial Sq^1 when $c = 0$ since all classes in the second column arise from cohomology of stabilizers of 2-cells in ${}_{\Gamma}\backslash X$ and the $\mathbb{F}_2$-dimensions of $E_2^{2,q}$ are equal for all q. However, by Property 5 in Section 2.2, $Sq^1 d_2 = d_2 Sq^1$, so this is impossible. On the other hand, when $c > 0$, some classes in $E_2^{2,1}$ will have trivial Sq^1 simply for dimensional reasons. These classes can be the target of a non-trivial d_2 differential. □

Remark 21. The $d_2^{0,2q}$ differential may vanish also in cases where $c \neq 0$. In the latter cases, we rely on the machine computations in order to find the rank of $d_2^{0,2q}$.

On the other hand, we have

Lemma 22. *The d_2 differential vanishes on $E_2^{0,4q}$.*

Proof. By periodicity, it is sufficient to show this in $E_2^{0,4}$. If d_2 does not vanish, then its image is in $E_2^{2,3}$ which is generated by a class in $\oplus\,\mathrm{H}^3(\mathbb{Z}/2)$, where the sum is over 2-cells in ${}_{\Gamma}\backslash X$. We follow the argument in Lemma 20, noting that since $\dim_{\mathbb{F}_2} E_2^{2,3} = \dim_{\mathbb{F}_2} E_2^{2,5}$, Sq^2 of any class in $E_2^{2,3}$ is non-zero. On the other hand, Sq^2 of the 4-dimensional polynomial class is always 0. Now Property 5 in Section 2.2 implies $Sq^2 d_2 = d_2 Sq^2$, which forces the vanishing result. □

The next few results relate to components of type $\circ\!\!\bullet$.

Lemma 23. [2, Lemma 33] *The d_2 differential is nontrivial on cohomology on components of type $\circ\!\!\bullet$ in degrees $q \equiv 1 \bmod 4$ if and only if it is nontrivial on these components in degrees $q \equiv 3 \bmod 4$.*

Lemma 24. *Let the non-central 2-torsion subcomplex X_s admit as quotient components of type $\circ\!\!\bullet$ only. Then the d_2 differential vanishes on $E_2^{0,2q}$.*

Proof. To prove this lemma, we will compare results from two different spectral sequences, the equivariant spectral sequence for the PSL_2 group (ESS), and the Lyndon-Hochschild-Serre spectral sequence (LH3S). We will distinguish the differentials in the spectral sequences via their superscripts. Let $\Gamma \subseteq \mathrm{SL}_2(\mathcal{O}_{-m})$ be a congruence subgroup and $\overline{\Gamma}$ be its image in $\mathrm{PSL}_2(\mathcal{O}_{-m})$. By the discussion after Definition 3, the reduced non-central 2-torsion subcomplex of Γ and reduced 2-torsion subcomplex of $\overline{\Gamma}$ are the same.

It is sufficient to show that the 2-dimensional polynomial class in $E_2^{0,2}$ from Theorem 8 is a permanent cocycle. Let Γ be a congruence subgroup where $\overline{\Gamma} \subseteq \mathrm{PSL}_2(\mathcal{O}_{-m})$ has a reduced 2-torsion subcomplex which consists solely of k components of type $\circ\!\!\bullet$. The E_2 page of the ESS for $\mathrm{H}^*(\overline{\Gamma})$ has the form

$$\begin{array}{c|lll}
3 & (\mathbb{F}_2)^k & (\mathbb{F}_2)^k & \\
2 & (\mathbb{F}_2)^k & (\mathbb{F}_2)^k & \\
1 & (\mathbb{F}_2)^k & (\mathbb{F}_2)^k & \\
0 & \mathbb{F}_2 & \oplus_{\beta^1}\mathbb{F}_2 & \oplus_{\beta^2}\mathbb{F}_2 \\
\hline
 & 0 & \qquad 1 & \qquad 2
\end{array}$$

where $\beta^q = \dim_{\mathbb{F}_2} \mathrm{H}^q({}_{\Gamma}\backslash X;\, \mathbb{F}_2)$. Once the d_2^{ESS} differential is determined in this spectral sequence, $E_3 = E_\infty$ and the calculation is complete. In any case, above dimension 2 (the virtual cohomological dimension), all classes in the 0 column of the ESS are polynomial by Property 3 in Section 2.2. In fact, we can say more. In $\mathrm{PSL}_2(\mathcal{O}_{-m})$, components of type $\bigcirc\!\bullet$ correspond to a graph of groups where there is a single vertex and single edge, both with $\mathbb{Z}/2$ stabilizer. At the level of presentations, this is the HNN extension $\langle t, x | x^2 = 1, t^{-1}xt = x\rangle$. In other words, in $\mathrm{PSL}_2(\mathcal{O}_{-m})$, $\mathrm{H}^*(\bigcirc\!\bullet) \cong \mathrm{H}^*(\mathbb{Z} \times \mathbb{Z}/2)$. Therefore, the classes in $E_2^{1,q}$ in the ESS for $\mathrm{H}^*(\overline{\Gamma})$ with $q > 0$ are products of an exterior class with a polynomial class.

To determine $\mathrm{H}^*(\Gamma)$ we use the Lyndon-Hochschild-Serre spectral sequence associated to the extension

$$1 \to \mathbb{Z}/2 \to \Gamma \to \overline{\Gamma} \to 1.$$

Analogous to the calculation of $\mathrm{H}^*(\mathbb{Z}/4)$ from $\mathrm{H}^*(\mathbb{Z}/2)$, it follows that $d_2^{\mathrm{LH3S}}(z_1) \neq 0$. To analyze the image of this differential, let z_1 be the polynomial generator of the cohomology ring of the central $\mathbb{Z}/2$ which forms the vertical edge of the LH3S. The horizontal edge of this spectral sequence can be identified with $\mathrm{H}^*(\overline{\Gamma})$ with untwisted coefficients. In addition, we note that in $\mathrm{SL}_2(\mathcal{O}_{-m})$, a $\bigcirc\!\bullet$ component is again a single edge and single vertex, this time with $\mathbb{Z}/4$ stabilizers. As a graph of groups, this corresponds to an HNN extension $\mathbb{Z}/4*_{\mathbb{Z}/4}$.

Assume that d_2^{LH3S} has a non-zero image in some $\bigcirc\!\bullet$ component of $\mathrm{H}^*(\overline{\Gamma})$. By direct calculation, one can show that if this component in the image of d_2^{LH3S} is the exterior class from $\mathrm{H}^*(\bigcirc\!\bullet)$, then the resulting cohomology ring is isomorphic to a polynomial ring over two 1-dimensional generators. This is not consistent with the calculation of $\mathrm{H}^*(\bigcirc\!\bullet) \cong \mathrm{H}^*(\mathbb{Z}/4*_{\mathbb{Z}/4})$ from Lemma 18, so we conclude that when the image of d_2^{LH3S} is non-zero in some $\bigcirc\!\bullet$ component, its image must be the polynomial class. That is, we may assume that the image is a class that arises from some place other than $E_2^{1,1}$ in the ESS calculation for $\mathrm{H}^*(\overline{\Gamma})$.

Moving on the E_3 page of the LH3S, we calculate using the compatibility between Sq^1 and the differential, applying the general version of Property 5 in Section 2.2 as given in [21, Theorem 2.17]. We have

$$d_3^{\mathrm{LH3S}}\left((z_1)^2\right) = d_3^{\mathrm{LH3S}}\left(Sq^1(z_1)\right) = Sq^1 d_2^{\mathrm{LH3S}}(z_1).$$

We claim that $Sq^1 d_2^{\mathrm{LH3S}}(z_1) = 0$. The summands in $d_2^{\mathrm{LH3S}}(z_1)$ are on the horizontal edge of the spectral sequence, hence they originate in $\mathrm{H}^2(\overline{\Gamma})$. From the analysis of the image of d_2^{LH3S} at the end of the prior paragraph, these classes originate from either $E_2^{0,2}$ or $E_2^{2,0}$ in the ESS calculation for $\mathrm{H}^*(\overline{\Gamma})$. For the summands that originate from the former, we note that $E_2^{0,2}$ is generated by squared 1-dimensional polynomial classes; these become zero under Sq^1. Next, we consider the latter case. Let σ be a summand in $E_2^{2,0}$ in ESS such that $Sq^1(\sigma)$ is a non-zero 3-dimensional class. We note from the ESS calculation that all products with σ are zero in $\mathrm{H}^*(\overline{\Gamma})$ by Property 2 in Section 2.2. Furthermore, the class $Sq^1(\sigma)$ has its origin in $E_2^{0,3}$ or $E_2^{1,2}$ in the ESS of $\mathrm{H}^*(\overline{\Gamma})$. Let x_1 be a polynomial generator that appears in the image of $Sq^1(\sigma)$. Then $Sq^1(x_1\sigma) = x_1 Sq^1(\sigma) \neq 0$. This is impossible as $x_1\sigma = 0$.

Summarizing, d_3^{LH3S} is the zero map on $E_2^{0,2}$. We conclude that this class, represented by z_1^2, is a permanent cocycle, and hence is a 2-dimensional polynomial class in $\mathrm{H}^*(\Gamma)$. □

Combining Lemmas 22, 23 and 24 with Theorem 8 and Lemma 18, we obtain

Corollary 25. *Let the non-central* 2*-torsion subcomplex* X_s *admit as quotient* $k \geq 1$ *components of type* $\bigcirc\!\bullet$*. Then*

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\Gamma; \mathbb{F}_2) = \begin{cases} \beta^1 + k - r, & q = 1, \\ \beta^2 + \beta^1 + k + c - r, & q \geq 2, \end{cases}$$

where $r := \operatorname{rank} d_2^{0,1}$, c *the co-rank of Section 5 and* $\beta^q = \dim_{\mathbb{F}_2} \mathrm{H}^q({}_{\Gamma}\backslash X; \mathbb{F}_2)$.

This result is similar to Lemmas 34 and 25 in [2]. We restate it here in the general setting of arbitrary reduced 2-torsion subcomplex components.

Lemma 26. *The* d_2 *differential is trivial in dimensions congruent to* $3 \bmod 4$ *on all non-central* 2*-torsion subcomplex components which are not of type* $\bigcirc\!\bullet$*.*

Proof. We note that by Theorem 5, the only non-trivial restriction map on cohomology in odd dimensions for finite subgroups is $res^{\mathbf{Q}_8}_{\mathbb{Z}/4}(x_1) = b_1$. In particular, the restriction maps on cohomology are zero in dimensions $4k+3$, so d_1 is trivial on these classes. Hence, these classes survive to the E_2 page. We note that Sq^2 is trivial on classes in both $\mathrm{H}^3(\mathbf{Te})$ and $\mathrm{H}^3(\mathbf{Q}_8)$. The former follows since $\mathrm{H}^5(\mathbf{Te}) = 0$; the latter since $Sq^2(x_1^2y_1) = x_1^4y_1 = 0$ by ring relations. On the other hand, $d_2 : E_2^{0,3} \to E_2^{2,2} \cong \bigoplus \mathrm{H}^2(\mathbb{Z}/2) \subset \bigoplus \mathrm{H}^*(\mathbb{Z}/2) \subset E_2^{2,*}$, where the sum is over the 2-dimensional cohomology classes of ${}_\Gamma\backslash X$. In the cohomology rings $\mathrm{H}^*(\mathbb{Z}/2)$ in this sum, $Sq^2(z_1^2) = z_1^4$. Since $d_2Sq^2 = Sq^2d_2$, we conclude that d_2 must vanish on $E_2^{0,3}$, and more generally on $E_2^{0,4k+3}$ by periodicity. $\square$

Combining Theorems 8 and 17, we obtain

Corollary 27. *Let the non-central 2-torsion subcomplex quotient ${}_\Gamma\backslash X_s$ consist of $k \geq 0$ components of type* $\circlearrowleft\!\bullet$*, and $j \geq 1$ connected components of equal type, but not* $\circlearrowleft\!\bullet$*. In one of the latter components, with the homotopy type of an l-looped rose, let m vertices have $\mathbf{Q}_8$ stabilizers, let n vertices have $\mathbf{Te}$ stabilizers. Then*

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\Gamma; \mathbb{F}_2) = \begin{cases} \beta^1 + jl + k - r^{0,1}, & q = 1, \\ \beta^2 + \beta^1 + j(m+n+l) - 1 + k + c - r^{0,1} - r^{0,2}, & q \equiv 2, \\ \beta^2 + \beta^1 + 2j(m+n) - 1 + k + c - r^{0,3} - r^{0,2}, & q \equiv 3, \\ \beta^2 + \beta^1 + j(m+n) + k + c - r^{0,3}, & q \equiv 4, \\ \beta^2 + \beta^1 + jl + k + c - r^{0,1}, & q \equiv 5, \end{cases}$$

where $r^{0,q}$ is the rank of the $d_2^{0,q}$-differential of the equivariant spectral sequence.

Note that for $k = 0$, Lemma 26 makes the formula of Corollary 27 boil down to

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\Gamma; \mathbb{F}_2) = \begin{cases} \beta^1 + jl - r^{0,1}, & q = 1, \\ \beta^2 + \beta^1 + j(m+n+l) - 1 + c - r^{0,1} - r^{0,2}, & q \equiv 2, \\ \beta^2 + \beta^1 + 2j(m+n) - 1 + c - r^{0,2}, & q \equiv 3, \\ \beta^2 + \beta^1 + j(m+n) + c, & q \equiv 4, \\ \beta^2 + \beta^1 + jl + c - r^{0,1}, & q \equiv 5. \end{cases}$$

We are not able to provide a result about the vanishing/non-vanishing of the d_2 differential in dimensions $\equiv 1 \bmod 4$. However, there is an alternative way to derive this information :
By the Universal Coefficient Theorem,

$$\mathrm{H}^1(\Gamma; \mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{H}_1(\Gamma; \mathbb{Z}), \mathbb{F}_2).$$

The rank of the right hand side is readily determined by finding the abelianization of Γ. By the table in Theorem 8, this rank is also equal to

$$\dim_{\mathbb{F}_2}(E_2^{0,1}(X_s)) + \dim_{\mathbb{F}_2} \mathrm{H}^1({}_\Gamma\backslash X) - r,$$

where r is the rank of the d_2 differential.

9. Ford Fundamental Domains

In this section, we construct a Ford fundamental domain for the congruence subgroup $\Gamma_0(\pi) < \mathrm{PSL}_2(\mathcal{O}_{-11})$ where $\langle \pi \rangle$ is a prime ideal in $\mathcal{O}_{-11} = \mathbb{Z}[\omega]$, the ring of integers of the number field $\mathbb{Q}[\sqrt{-11}]$, and $\omega = (-1+\sqrt{-11})/2$. We prove that, given a prime ideal $\langle \pi \rangle \subset \mathcal{O}_{-11}$, there exists a Ford domain for the congruence subgroup $\Gamma_0(\pi) < \mathrm{PSL}_2(\mathcal{O}_{-11})$ which has a particular structure similar to that found by Orive [13] for congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. Similar Ford domains may be found for congruence subgroups of this form in other Bianchi groups. We first recall the definition of such a fundamental domain.

Given $\gamma \in \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}_2(\mathbb{C})$, the *isometric sphere* S_γ of γ is defined to be the (hemi)sphere on which γ acts as a Euclidean isometry; it can be shown that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then S_γ has radius $1/|c|$ and center $-d/c \in \mathbb{C}$. The isometric sphere $S_{\gamma^{-1}}$ of the inverse γ^{-1} has the same radius and center a/c. A *Ford fundamental domain* F for Γ is then the intersection between the region $B \subset \mathbb{H}^3$, exterior to all isometric spheres, and a fundamental domain F_∞ for the subgroup $\Gamma_\infty < \Gamma$ of elements which fix ∞.

We shall call *visible spheres* those isometric spheres which contribute to the boundary of B, because they are the spheres one "sees" when viewing the boundary of B from above; beneath them are infinitely many smaller spheres, each completely covered by the collection of spheres which bound B. We say that an isometric sphere S *covers* a point $p \in \mathbb{C}$ if p lies in the interior of S, and we say that the sphere S is covered by other spheres $\{S_i\}$ if the union of the interiors of the S_i contains the interior of S.

We note that a Ford domain for $\mathrm{PSL}_2(\mathcal{O}_{-11})$ is constructed as follows. The region B exterior to all isometric spheres is bounded by those isometric spheres of radius 1 centered at points of the ring $\mathcal{O}_{-11}$. The fundamental domain F_∞ can be taken to have vertices at

$$\begin{gathered}\pm z_1 = \pm(3/11 + 6\omega/11) = \pm 3i/\sqrt{11},\\ \pm z_2 = \pm(-3/11 + 5\omega/11) = \pm(-1/2 + 5i/(2\sqrt{11})) \text{ and}\\ \pm z_3 = \pm(8/11 + 5\omega/11) = \pm(1/2 + 5i/(2\sqrt{11})),\end{gathered}$$

and thus the Ford domain is the convex hyperbolic polyhedron with vertices at the cusp ∞ and above these six points of $\mathbb{C}$ at height $\sqrt{2/11}$.

In the following, we will appeal to the following Lemmas about isometric spheres of elements of $\mathrm{PSL}_2(\mathcal{O}_{-11})$.

Lemma 28. *The only elements of* $\mathrm{PSL}_2(\mathcal{O}_{-11})$ *whose isometric spheres cover* 0 *are those of the form* $\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ *for* $a \in \mathcal{O}_{-11}$.

Proof. In order for an isometric sphere to cover 0, the radius of the sphere must be strictly larger than the complex modulus of its center. For an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_{-11})$, the radius is $1/|c|$ and the complex modulus of the center is $|-d/c| = |d|/|c|$. Therefore, the isometric sphere covers 0 if and only if we have $1 > |d|$ for $d \in \mathcal{O}_{-11}$, and hence we must have $d = 0$. The requirement that the determinant be 1 then means that we have $ad - bc = -bc = 1$. The only non-zero elements of $\mathcal{O}_{-11}$ of modulus at most 1 are ± 1, and so we see that $b = -c = \pm 1$. Since we are working in $\mathrm{PSL}_2(\mathcal{O}_{-11})$, we may choose $c = 1$. □

As a consequence of Lemma 28, we see that there is only one isometric sphere covering 0, and it is the sphere of radius 1 centered at the cusp 0.

Lemma 29. *If the isometric sphere of radius 1 centered at the cusp* 0 *is removed from the Ford domain for* $\mathrm{PSL}_2(\mathcal{O}_{-11})$*, the isometric spheres which become visible are those corresponding to the elements*

$$\begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega+1 & 1 \end{pmatrix},$$

and their inverses.

Proof. To see this, consider the isometric spheres of radius 1 centered at elements of $\mathcal{O}_{-11}$, and consider removing the sphere centered at the cusp 0. We then see the isometric spheres of

$$\begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \omega+1 & 1 \end{pmatrix},$$

and their inverses, visible between the spheres of radius 1 centered at ± 1, $\pm\omega$ and $\pm\overline{\omega}$. These smaller spheres are centered at $\pm 1/6 \pm \sqrt{-11}/6$ and have radius $1/\sqrt{3}$. Together with the radius 1 spheres centered at ± 1, these six spheres all intersect at the cusp 0, and in vertical circles of radius $\sqrt{11}/6$. The smaller spheres intersect the radius 1 spheres centered at $\pm\omega$ and $\pm\overline{\omega}$ in the same locus which the removed sphere did.

By Lemma 28, no other sphere can cover 0. Another visible sphere which intersects 0 must have radius at least $\sqrt{11}/6$. But since this radius is larger than $1/2$, the only possibilities are that the radius could be 1 or $1/\sqrt{3}$, and these matrices have already been considered. □

In the following, we will denote by $\mathcal{B}$ the collection of isometric spheres of radius 1 centered at all elements of $\mathcal{O}_{-11}$. Then, given an ideal $\langle\pi\rangle \subset \mathcal{O}_{-11}$, we will denote by $\mathcal{B}'_{\langle\pi\rangle}$ the collection of isometric spheres $\mathcal{B}$ with those centered at elements of the ideal $\langle\pi\rangle$ removed. The spheres which then become visible are described by Lemma 29.

Let $p \in \mathbb{Z}$ be an odd rational prime.

9.1. **Case 1: p splits in $\mathcal{O}_{-11}$.** Suppose that $p = \pi\overline{\pi}$, where $\pi = a + b\omega$. We suppose that $a \geq 0$ and that $b > 0$. We will find a Ford domain for the subgroup $\Gamma_0(\pi)$. We first establish which isometric spheres form the boundary of the set B.

For each $\alpha \in \mathcal{O}_{-11} \setminus \langle\pi\rangle$ there is a corresponding element

$$\begin{pmatrix} * & * \\ \pi & -\alpha \end{pmatrix}$$

and for no $\alpha \in \langle\pi\rangle$ is there a similar element, as this would cause the determinant to be divisible by π. The isometric sphere of the given element has center α/π and radius $1/\sqrt{p}$. Consider the collection of spheres $\mathcal{S}_{\langle\pi\rangle}$ of radius $1/\sqrt{p}$ centered at α/π for $\alpha \in \mathcal{O}_{-11} \setminus \langle\pi\rangle$. Include also in $\mathcal{S}_{\langle\pi\rangle}$ the isometric spheres of

$$\begin{pmatrix} 1 & 0 \\ \omega\pi & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ (\omega+1)\pi & 1 \end{pmatrix},$$

their inverses, and the translates of these spheres by Γ_∞; these spheres have radius $1/\sqrt{3p}$. We claim that the spheres of $\mathcal{S}_{\langle\pi\rangle}$ suffice to determine the boundary of B.

Suppose for sake of contradiction that there exists an element

$$M = \begin{pmatrix} \alpha & \beta \\ \delta\pi & \gamma \end{pmatrix}$$

where $\delta \neq 1$, whose isometric sphere is visible above $\mathcal{S}_{\langle\pi\rangle}$. Then we may apply an isometry

$$\psi_\pi = \begin{pmatrix} \sqrt{\pi} & 0 \\ 0 & 1/\sqrt{\pi} \end{pmatrix}$$

of $\mathbb{H}^3$ which moves $\mathcal{S}_{\langle\pi\rangle}$ to $\mathcal{B}'_{\langle\pi\rangle}$ and conjugates M to

$$M' = \psi_\pi M \psi_\pi^{-1} = \begin{pmatrix} \alpha & \beta\pi \\ \delta & \gamma \end{pmatrix}.$$

By the assumption, the isometric sphere of M' is visible above those of $\mathcal{B}'_{\langle\pi\rangle}$. But by Lemma 29, no such spheres are visible, and we have a contradiction.

We now have a complete list of visible isometric spheres. We note that the vertices at which these spheres intersect are exactly those of $\mathcal{B}'_{\langle\pi\rangle}$ with the isometry ψ_π^{-1} applied to them; the vertices of $\mathcal{B}'_{\langle\pi\rangle}$ are the vertices at $\pm z_1$, $\pm z_2$, $\pm z_3$, and their translates by Γ_∞, and at height $\sqrt{2/11}$, where z_1, z_2 and z_3 were defined above. After applying ψ_π^{-1}, these vertices are located at $(\pm z_1 + \alpha)/\pi$, $(\pm z_2 + \alpha)/\pi$, and $(\pm z_3 + \alpha)/\pi$, for $\alpha \in \mathcal{O}_{-11}$, at height $\sqrt{2/11p}$.

It remains to choose a fundamental domain for the action of Γ_∞. We claim that each isometric sphere of radius $1/\sqrt{p}$ is Γ_∞-equivalent to one centered at a point c/π for $c \in \mathbb{Z} \setminus (p)$, where $(p) \subset \mathbb{Z}$ is the prime ideal generated by p. To see this, we wish to show that there exist $l, m \in \mathbb{Z}$ such that $\alpha/\pi + l + m\omega = c/\pi$ for $c \in \mathbb{Z}$. Writing $\alpha = x + y\omega$, we have

$$\frac{\alpha}{\pi} + l + m\omega = \frac{x + y\omega + (l + m\omega)\pi}{\pi}.$$

Writing $\pi = a + b\omega$,

$$\frac{x + y\omega + (l+m\omega)\pi}{\pi} = \frac{x + y\omega + (l+m\omega)(a+b\omega)}{\pi} = \frac{x + y\omega + al + am\omega + bl\omega + bm\omega^2}{\pi}.$$

Writing $\omega^2 = -3-\omega$,

$$\frac{x+y\omega+al+am\omega+bl\omega+bm\omega^2}{\pi} = \frac{x+y\omega+al+am\omega+bl\omega-3bm-bm\omega}{\pi} = \frac{x+al-3bm+(y+am+bl-bm)\omega}{\pi}.$$

Since $a,b,l,m,x,y \in \mathbb{Z}$, the numerator is a rational integer if and only if

$$y+am+bl-bm = y+bl+(a-b)m = 0.$$

But since a and b are necessarily relatively prime, it follows that so are b and $a-b$, and so we may choose rational integers l and m so that $bl+(a-b)m = -y$.

We next note that none of the isometric spheres centered at j/π, for $1 \le j \le p-1$, are Γ_∞-equivalent, and that for all $j \in \mathbb{Z}$, j/π is Γ_∞-equivalent to $(j+p)/\pi = (j/\pi)+\overline{\pi}$. Thus every visible sphere of $\mathcal{S}_{\langle\pi\rangle}$ is Γ_∞-equivalent to one of those centered at j/π, $1 \le j \le p-1$, or to one of those of radius $1/\sqrt{3p}$ which intersect at the cusp 0. It is most convenient to have our fundamental domain have its faces on the isometric spheres centered at j/π for $-(p-1)/2 \le j \le (p-1)/2$ (and $j \ne 0$).

We therefore take as our fundamental domain for Γ_∞ the region above these faces. This is the convex hull of the cusps 0, ∞ and the vertices at $\pm(z_1+j)/\pi$, $\pm(z_2+j)/\pi$, and $\pm(z_3+j)/\pi$ for $-(p-1)/2 \le j \le (p-1)/2$ at height $\sqrt{2/(11p)}$.

9.2. **Case 2: p ramifies in $\mathcal{O}_{-11}$.** In this case, we have $p = 11$, and we will find a Ford domain for the group $\Gamma_0(\pi)$ where $\pi = \sqrt{-11} = 1+2\omega$. This works the same as Case 1, with $a = 1$ and $b = 2$. Applying the method described above, we find a Ford domain bounded by isometric spheres of radius $1/\sqrt{11}$ and centered at $j/\sqrt{-11}$ for $j = \pm1, \pm2, \pm3, \pm4$, and ±5, along with four spheres of radius $1/\sqrt{33}$ centered at $\pm1/(\omega\sqrt{-11})$ and $\pm1/((\omega+1)\sqrt{-11})$. The vertices of this domain are $\pm(z_1+j)/\sqrt{-11}$, $\pm(z_2+j)/\sqrt{-11}$, and $\pm(z_3+j)/\sqrt{-11}$ for $-5 \le j \le 5$ at height $\sqrt{2/121} = \sqrt{2}/11$.

9.3. **Case 3: p is inert in $\mathcal{O}_{-11}$.** We will find a Ford domain for the group $\Gamma_0(p)$. We first determine the isometric spheres which contribute to B.

For each element $\alpha \in \mathcal{O}_{-11} \setminus \langle p\rangle$, there is a sphere of B corresponding to the element

$$\begin{pmatrix} * & * \\ p & -\alpha \end{pmatrix}.$$

We therefore have spheres of radius $1/p$ centered at all points α/p for $\alpha \in \mathcal{O}_{-11}$, with those spheres centered at $(ap+bp\omega)/p = a+b\omega$ for $a,b \in \mathbb{Z}$ (i.e., those centered at points of $\mathcal{O}_{-11}$) removed. Consider the collection of spheres $\mathcal{S}_{\langle p\rangle}$ which consists of these spheres and of the isometric spheres of the elements

$$\begin{pmatrix} 1 & 0 \\ p\omega & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ p(\omega+1) & 1 \end{pmatrix},$$

their inverses, and the translates of these spheres by Γ_∞. We claim that the spheres of $\mathcal{S}_{\langle p\rangle}$ suffice to determine the boundary of B.

To see this, suppose that another isometric sphere, belonging to an element

$$M = \begin{pmatrix} \alpha & \beta \\ \delta p & \gamma \end{pmatrix},$$

is visible above those of $\mathcal{S}_{\langle p\rangle}$. We conjugate by the element

$$\psi_p = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 1/\sqrt{p} \end{pmatrix}$$

which has the effect of moving $\mathcal{S}_{\langle p\rangle}$ to $\mathcal{B}'_{\langle p\rangle}$. As such, the isometric sphere of the element

$$M' = \psi_p M \psi_p^{-1} = \begin{pmatrix} \alpha & \beta p \\ \delta & \gamma \end{pmatrix}$$

is visible above those of $\mathcal{B}'_{\langle p\rangle}$. But by Lemma 29, no such spheres are visible, and we have a contradiction.

We now have a complete list of visible isometric spheres. We note that the vertices at which these spheres intersect are exactly those of $\mathcal{B}'_{\langle p\rangle}$ with the isometry ψ_p^{-1} applied to them; the vertices of $\mathcal{B}'_{\langle p\rangle}$ are the vertices at $\pm z_1$, $\pm z_2$, $\pm z_3$, and their translates by Γ_∞, and at height $\sqrt{2/11}$. After applying ψ_p^{-1}, these vertices are located at

$$(\pm z_1+\alpha)/p,\ (\pm z_2+\alpha)/p, \text{ and } (\pm z_3+\alpha)/p, \text{ for } \alpha\in\mathcal{O}_{-11}, \text{ at height } \sqrt{2/(11p^2)}.$$

It remains to choose a fundamental domain for the action of Γ_∞. Each of the visible spheres of radius $1/(p\sqrt{3})$ is Γ_∞-equivalent to one of those which intersects 0. Each sphere of radius $1/p$ is Γ_∞-equivalent to one of those centered at $j/p+k\omega/p$ for j,k integers with $-(p-1)/2\leq j,k\leq(p-1)/2$ and j,k not both 0. Since these spheres are not pairwise Γ_∞-equivalent, we may take the vertices where these spheres intersect (including vertices where they intersect spheres adjacent to these spheres) as the vertices of our fundamental domain. The vertices are therefore located at $(z_1+j+k\omega)/p$, $(-z_1+j+k\omega)/p$, $(z_2+j+k\omega)/p$, $(-z_2+j+k\omega)/p$, $(z_3+j+k\omega)/p$, and $(-z_3+j+k\omega)/p$, for $-(p-1)/2\leq j,k\leq(p-1)/2$. These vertices are located at height $\sqrt{2/(11p^2)}$. Our fundamental domain is the convex hull of these vertices and the cusps 0 and ∞.

9.4. **Other Bianchi Groups.** A similar method produces Ford fundamental domains for congruence subgroups $\Gamma_0(\pi)$ of prime level in the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$, where $\mathcal{O}_{-m}$ is another Euclidean ring, i.e. the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-m}\,)$, for $m=1,2,3,7$. The argument above may be adapted to show that the set B is bounded by isometric spheres of radius $1/|\pi|$ centered at a/π for $a\in\mathcal{O}_{-m}\setminus\langle\pi\rangle$, and by at most six smaller isometric spheres, which meet at the cusp 0, and their translates by Γ_∞. Furthermore, the fundamental domain for the action of Γ_∞ may be chosen so that the resulting fundamental domain is bounded by $N(\pi)-1$ spheres of radius $1/|\pi|$ (where $N(\pi)$ denotes the norm of π in $\mathcal{O}_{-m}$), by the smaller spheres which meet at the cusp 0, and by the vertical sides which make up the boundary of the fundamental domain of the action of Γ_∞.

The fundamental domains constructed in this section for $\Gamma_0(\pi)$ are composed of $N(\pi)+1$ copies of a Ford domain for the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_{-m})$, where $N(\pi)+1=[\mathrm{PSL}_2(\mathcal{O}_{-m}):\Gamma_0(\pi)]$. We believe that it should be possible to construct fundamental domains for congruence subgroups of prime level in the other Bianchi groups in a similar way. Furthermore, it should be possible to construct Ford domains for composite levels from copies of those for prime levels in a fashion akin to that described by Orive [13] in the case of $\mathrm{PSL}_2(\mathbb{Z})$.

10. Example computations

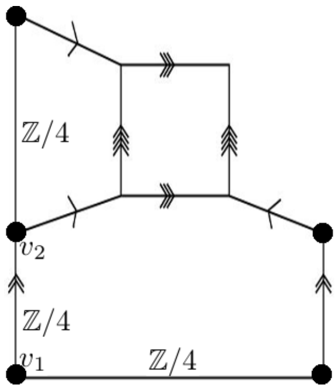
We print the details only for two of our example calculations. Checking the other outcomes of example calculations mentioned in Section 4 is made straightforward by the algorithm of Section 9.

10.1. **Level $\sqrt{-2}$ in $\mathbf{SL}_2(\mathbb{Z}[\sqrt{-2}])$.** Let

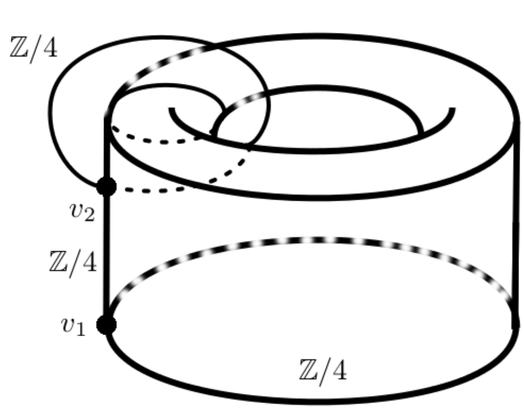
$$\Gamma:=\Gamma_0(\sqrt{-2}):=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])\;\middle|\; c\in\langle\sqrt{-2}\rangle\right\}.$$

A fundamental domain for Γ is given in Figure 1a. Then, the orbit space including the non-central 2–torsion subcomplex is drawn in Figure 1b. Hence ${}_{\Gamma}\backslash X_s=$ , $v=2$, $\chi({}_{\Gamma}\backslash X_s)=-1$. Using Figure 1b, we convince ourselves that $c=0$, $\beta^1(({}_{\Gamma}\backslash X)=2$ and $\beta^2(({}_{\Gamma}\backslash X)=1$. From Theorem 17, we see that the non-central 2–torsion subcomplex contributes the following dimensions to the E_2 page.

$q\equiv 3 \mod 4$	$(\mathbb{F}_2)^2$	$(\mathbb{F}_2)^3$
$q\equiv 2 \mod 4$	$(\mathbb{F}_2)^4$	$(\mathbb{F}_2)^3$
$q\equiv 1 \mod 4$	$(\mathbb{F}_2)^2$	$\mathbb{F}_2$
$q\equiv 0 \mod 4$	$\mathbb{F}_2$	$(\mathbb{F}_2)^2$
	$p=0$	$p=1$



(A) Strict fundamental domain for the 2-dimensional equivariant retract (away from the principal ideal cusps), extracted from a Ford fundamental domain for $\Gamma_0(\sqrt{-2})$, after the Borel–Serre bordification, replacing the cusps by Euclidean planes with wallpaper group action (hence a rectangle with identified sides in the fundamental domain).



(B) Orbit space of the displayed fundamental domain for $\Gamma_0(\sqrt{-2})$, with the cusp having become a 2-torus in the Borel–Serre compactification. The 2–torsion subcomplex is given by the edges labelled by $\mathbb{Z}/4$, and their vertices.

FIGURE 1. Fundamental domain for $\Gamma_0(\sqrt{-2})$ in $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$

From Lemma 26 and applying the method described just below it, we know that the d_2 differential of the equivariant spectral sequence vanishes. Applying Theorem 8, this allows us to conclude that the dimensions of the cohomology ring that we are looking for, are

$$\dim_{\mathbb{F}_2} \mathrm{H}^q(\Gamma_0(\sqrt{-2});\, \mathbb{F}_2) = \begin{cases} c + \beta^1({}_\Gamma\backslash X) + \beta^2({}_\Gamma\backslash X) + 2, & q = 4k+5, \\ c + \beta^1({}_\Gamma\backslash X) + \beta^2({}_\Gamma\backslash X) + 2, & q = 4k+4, \\ c + \beta^1({}_\Gamma\backslash X) + \beta^2({}_\Gamma\backslash X) + 3, & q = 4k+3, \\ c + \beta^1({}_\Gamma\backslash X) + \beta^2({}_\Gamma\backslash X) + 3, & q = 4k+2, \\ \beta^1({}_\Gamma\backslash X) + 2, & q = 1 \end{cases} = \begin{cases} 5, & q = 4k+5, \\ 5, & q = 4k+4, \\ 6, & q = 4k+3, \\ 6, & q = 4k+2, \\ 4, & q = 1. \end{cases}$$

10.2. **Level 5 in $\mathbf{SL}_2(\mathbb{Z}[\sqrt{-2}])$.** Let

$$\Gamma := \Gamma_0(5) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}]) \;\middle|\; c \in \langle 5 \rangle \right\}.$$

With the algorithm described in Section 9, we produce the Ford fundamental domain displayed in Figure 2a. Carrying out the side idenfications, under which each face of the Ford domain has a conjugate face, we are left with the strict (in its interior) fundamental domain of Figure 2b, which is subject to the indicated edge identifications. Then the 2-cells boundary matrix ∂_2 has elementary divisors

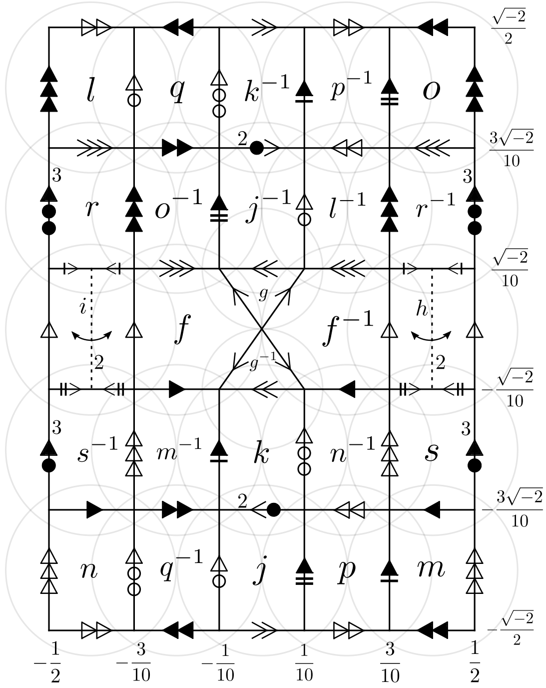
- 1, of multiplicity 12
- 2 and 4, each of multiplicity 1.

Its kernel is one-dimensional. The edges boundary matrix ∂_1 has the only elementary divisor 1, of multiplicity 6. Therefore, the cellular chain complex of the orbit space,

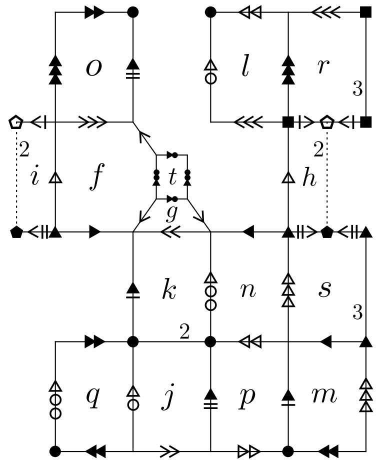
$$\mathbb{Z}^{15} \quad -\partial_2 \to \quad \mathbb{Z}^{22} \quad -\partial_1 \to \quad \mathbb{Z}^7,$$

has homology $\mathrm{H}_2({}_\Gamma\backslash X,\, \mathbb{Z}) \cong \mathbb{Z}$, $\quad \mathrm{H}_1({}_\Gamma\backslash X,\, \mathbb{Z}) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, $\quad \mathrm{H}_0({}_\Gamma\backslash X,\, \mathbb{Z}) \cong \mathbb{Z}$. Hence the remaining parameters for Theorem 8 are $\beta^1 = 4$, $\beta^2 = 3$, $a_3 = 4$, $a_2 = 3 + c$ and $a_1 = 3 + c$.

From Figure 2b, we see that ${}_\Gamma\backslash X_s \cong \circ\!\bullet\!\circ$, so $\chi({}_\Gamma\backslash X_s) = 0 = v$; furthermore for the quotient of the 3-torsion subcomplex, we also read off the type $\circ\!\bullet\!\circ$. Because of non-adjacency of the torus 2-cell to the torsion subcomplexes, the $d^2_{2,0}$-differential of the equivariant spectral sequence converging to $\mathrm{H}_{p+q}(P\Gamma,\, \mathbb{Z})$ vanishes; and therefore this spectral sequence yields the short exact



(A) Bottom facets of a Ford fundamental domain for $\Gamma_0(5)$.



(B) Strict fundamental domain for the 2-dimensional equivariant retract (away from the principal ideal cusps) extracted from the displayed Ford fundamental domain for $\Gamma_0(5)$, with the cusp having been replaced by a rectangle in the Borel–Serre bordification (passing to a 2-torus in the Borel–Serre compactification). The 2–torsion subcomplex is given by the edges labelled by a digit 2, and their vertices.

FIGURE 2. Ford fundamental domain for $\Gamma_0(5)$ in $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$

sequence

$$1 \to (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \to \mathrm{H}_1(P\Gamma,\, \mathbb{Z}) \to \mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4 \to 1,$$

which is compatible with the Abelianization of the presentation for $P\Gamma$ produced with the algorithm of Section 9, as well as with Schönnenbeck's machine result

$$\mathrm{H}_1(P\Gamma,\, \mathbb{Z}) \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/12)^2 \oplus \mathbb{Z}/2.$$

We now apply Theorem 8, and get $\dim \mathrm{H}^1(\Gamma,\, \mathbb{F}_2) = 6 - \mathrm{rank} d_2^{0,1}$. Using the Abelianization that we get from the fundamental domain in Figure 2a, or alternatively Schönnenbeck's machine computation described in the appendix, we infer $\mathrm{rank} d_2^{0,1} = 1$.

Appendix A. Machine computations

We use Sebastian Schönnenbeck's implementation [19] of the Voronoï cell complex to compute the cohomology of a sample (ten imaginary quadratic fields, six levels η in each of them) of congruence subgroups $\Gamma_0(\eta)$. Then we use Bui Anh Tuan's implementation of Rigid Facets Subdivision in order to extract the non-central 2-torsion subcomplex. This allows us to check our example computations with the algorithm of Section 9, and to illustrate which values the parameters in our formulas can take.

We let $K := \mathbb{Q}(\sqrt{-m}\,)$ and denote its discriminant by Δ. We present the ideal η in the ring of integers of K with the smallest possible number of generators; hence when we use two generators, it is because η is not principal. We let c be the co-rank defined in Section 5; let $\beta_q = \dim_{\mathbb{Q}} \mathrm{H}_q({}_{\Gamma_0(\eta)}\backslash\mathcal{H};\, \mathbb{Q})$ and $\beta^q = \dim_{\mathbb{F}_2} \mathrm{H}^q({}_{\Gamma_0(\eta)}\backslash\mathcal{H};\, \mathbb{F}_2)$ for $q = 1, 2$; and ${}_{\Gamma_0(\eta)}\backslash X_s$ the orbit space of the non-central 2-torsion subcomplex. We further write $H^q := \dim_{\mathbb{F}_2} \mathrm{H}^q(\Gamma_0(\eta);\, \mathbb{F}_2)$. Let r be the rank of the $d_2^{0,1}$-differential of the equivariant spectral sequence. In many cases, notably for $c = 0$, or when there are no components different from $\circ\!\!\bullet$ in ${}_{\Gamma_0(\eta)}\backslash X_s$, the $d_2^{0,2+4k}$-differential vanishes because of the lemmata in Section 8. *Note that for one case in our sample,* $\Gamma_0(\langle 2, \sqrt{-6}\rangle)$ *in* $SL_2(\mathbb{Z}[\sqrt{-6}\,])$*, the machine computation yields rank*$(d_2^{0,2+4k}) = 1$. In all other cases in our sample, the computation yields $d_2^{0,2+4k} = 0$, so we do not print $d_2^{0,2+4k}$. The machine calculations in HAP [7] allow to produce H^q and β_1 directly from Sebastian Schönnenbeck's cell complexes. To get β^1, r and c in the below table, we conclude from H^q and β_1 using the corollaries in Section 8. When ${}_{\Gamma_0(\eta)}\backslash X_s$ is empty, then because of Proposition 19, only the Betti numbers are of interest, so we put those cases in a separate table below. Recall that in all cases except for the Eisenstein integers in $\mathbb{Q}(\sqrt{-3}\,)$, the Euler characteristic χ of ${}_{\Gamma_0(\eta)}\backslash\mathcal{H}$ vanishes, so we only need β^1. Therefore, we indicate $\chi = 1 - \beta_1 + \beta_2$ only in those Eisenstein integers cases, where it is non-zero.

Δ	$-m$	Level η	$\Gamma_0(\eta)\backslash X_s$	β_1	χ	β^1	r	c	H^1	H^2	H^3	H^4	H^5
-3	-3	$\langle 2\rangle$		0	2	0	0	0	0	2	4	3	1
-11	-11	$\langle 2\rangle$		2		2	0	0	2	4	6	5	3
-15	-15	$\langle 2, \frac{3+\sqrt{-m}}{2}\rangle$		4		4	0	0	5	8	8	8	8
-7	-7	$\langle \frac{\sqrt{-m}-1}{2}\rangle$		2		2	0	0	3	4	4	4	4
-15	-15	$\langle \frac{\sqrt{-m}-5}{2}\rangle$		8		$8+r$	$r \leq 1$	$1-r$	10	18	18	18	18
-7	-7	$\langle 3\frac{\sqrt{-m}-1}{2}\rangle$		4		$4+r$	$r \leq 1$	$1-r$	6	10	10	10	10
-8	-2	$\langle 5\rangle$		2		$3+r$	$r \leq 2$	$2-r$	5	9	9	9	9
-8	-2	$\langle 3+2\sqrt{-m}\rangle$		2		$2+r$	$r \leq 2$	$2-r$	4	7	7	7	7
-56	-14	$\langle 2, \sqrt{-m}\,\rangle$		10		10	1	0	12	21	21	21	21
-56	-14	$\langle 2\rangle$		16		16	0	0	19	34	34	34	34
-52	-13	$3\langle 2, 1+\sqrt{-m}\,\rangle$		25		$30+r$	$r \leq 3$	$3-r$	33	65	65	65	65
-40	-10	$\langle 2+\sqrt{-m}\,\rangle$		10		$10+r$	$r \leq 2$	$2-r$	13	24	24	24	24
-40	-10	$\langle 5\rangle$		25		$26+r$	$r \leq 3$	$3-r$	29	57	57	57	57
-40	-10	$\langle 5, \sqrt{-m}\,\rangle$		5		$6+r$	$r \leq 3$	$3-r$	9	17	17	17	17
-24	-6	$\langle 2\rangle$		7		7	0	0	10	16	16	16	16
-20	-5	$\langle 10, 5+\sqrt{-m}\,\rangle$		8		$8+r$	$r \leq 3$	$3-r$	11	21	21	21	21
-40	-10	$\langle \sqrt{-m}\,\rangle$	5	10		$12+r$	$r \leq 5$	$5-r$	17	33	33	33	33
-24	-6	$\langle 2+\sqrt{-m}\,\rangle$	5	8		$8+r$	$r \leq 3$	$3-r$	13	23	23	23	23
-56	-14	$\langle 10, 4+\sqrt{-m}\,\rangle$	6	18		$18+r$	$r \leq 5$	$5-r$	24	45	45	45	45
-40	-10	$\langle 2, \sqrt{-m}\,\rangle$		5		$5+r$	$r \leq 1$	$1-r$	8	14	14	13	13
-24	-6	$\langle 2, \sqrt{-m}\,\rangle$		4		4	0	1	7	11	11	11	11
-52	-13	$\langle 2, 1+\sqrt{-m}\,\rangle$		7		7	0	0	9	16	16	15	15
-20	-5	$\langle 2, 1+\sqrt{-m}\,\rangle$		4		4	0	0	6	10	10	9	9
-8	-2	$\langle 2\rangle$		3		3	0	0	5	8	8	7	7
-52	-13	$\langle 2\rangle$		12		12	0	0	16	30	30	27	27
-40	-10	$\langle 2\rangle$		9		9	0	0	13	24	24	21	21
-20	-5	$\langle 2\rangle$		6		6	0	0	10	18	18	15	15
-8	-2	$\langle \sqrt{-m}\,\rangle$		2		2	0	0	4	6	6	5	5

TABLE 1. Cases $\Gamma_0(\eta)$ in the sample with non-empty non-central 2-torsion subcomplex

Δ	-56	-56	-56	-52	-52	-52	-24
$-m$	-14	-14	-14	-13	-13	-13	-6
η	$\langle 3, 1+\sqrt{-m}\rangle$	$\langle 6, 2+\sqrt{-m}\rangle$	$\langle 9, 2+\sqrt{-m}\rangle$	$\langle 7, 1+\sqrt{-m}\rangle$	$\langle \sqrt{-m}-1\rangle$	$\langle \sqrt{-m}+6\rangle$	$\langle 3, \sqrt{-m}\rangle$
β^1	10	21	20	6	14	20	5

Δ	-24	-24	-20	-20	-20	-15	-15	-15	-15
$-m$	-6	-6	-5	-5	-5	-15	-15	-15	-15
η	$\langle \sqrt{-m}\rangle$	$\langle 3\rangle$	$\langle 3, 1+\sqrt{-m}\rangle$	$\langle \sqrt{-m}+1\rangle$	$\langle \sqrt{-m}-2\rangle$	$\langle 3, \sqrt{-m}\rangle$	$\langle \frac{\sqrt{-m}-1}{2}\rangle$	$\langle \frac{\sqrt{-m}+3}{2}\rangle$	$\langle 3\rangle$
β^1	10	11	4	8	8	4	6	8	8

Δ	-11	-11	-11	-11	-11	-8	-8
$-m$	-11	-11	-11	-11	-11	-2	-2
η	$\langle \frac{\sqrt{-m}-1}{2}\rangle$	$\langle 4\rangle$	$\langle 1-\sqrt{-m}\rangle$	$\langle \frac{\sqrt{-m}+5}{2}\rangle$	$\langle \sqrt{-m}+2\rangle$	$\langle \sqrt{-m}-1\rangle$	$\langle \sqrt{-m}+2\rangle$
β^1	2	6	4	4	5	2	4

Δ	-7	-7	-7	-7	-3	-3	-3	-3	-3
$-m$	-7	-7	-7	-7	-3	-3	-3	-3	-3
η	$\langle \frac{\sqrt{-m}+3}{2}\rangle$	$\langle \sqrt{-m}\rangle$	$\langle \frac{7+\sqrt{-m}}{2}\rangle$	$\langle 7\rangle$	$\langle \sqrt{-m}\rangle$	$\langle 4\rangle$	$\langle 2\sqrt{-m}\rangle$	$\langle 3\rangle$	$\langle \frac{\sqrt{-m}-9}{2}\rangle$
β^1	3	2	4	8	0	1	0	0	1
χ					2	2	4	4	4

TABLE 2. Cases $\Gamma_0(\eta)$ in the sample with empty non-central 2-torsion subcomplex

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