CHEEGER CONSTANTS OF HYPERBOLIC REFLECTION GROUPS AND MAASS CUSP FORMS OF SMALL EIGENVALUES

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Abstract. We compute the Cheeger constants of a collection of hyperbolic surfaces corresponding to maximal non-compact arithmetic Fuchsian groups, and to subgroups which are the rotation subgroup of maximal reflection groups. The Cheeger constants are geometric quantities, but relate to the smallest eigenvalues of Maass cusp forms. From geometrical considerations, we find evidence for the existence of small eigenvalues. We search for these small eigenvalues and compute the corresponding Maass cusp forms numerically.

1. Introduction

Given a hyperbolic Riemann surface $M = \Gamma\backslash\mathbb{H}^2$ for a cofinite, discrete Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$, the *Cheeger constant* of M is defined as

$$h(M) = \inf_{E\subset M} \frac{\ell(E)}{\min\{\mathrm{Area}(A), \mathrm{Area}(B)\}}$$

where E varies over all 1-dimensional subsets of M which divide M into two disjoint components A and B, and $\ell(E)$ denotes the length of E. This quantity is related to the geometry of M and was introduced by Cheeger [8]. The quantity is also related to the first eigenvalue λ_1 of the Laplacian operator on M; when M is a closed Riemannian manifold (i.e., M has no cusps or cone points, or equivalently Γ has no parabolic or elliptic elements), h and λ_1 are related via the inequalities

$$\frac{h^2(M)}{4} \leq \lambda_1(M) \leq 2h(M) + 10h^2(M),$$

where the first inequality is due to Cheeger [8], and the second to Buser [7]. When M is finite area, but permitted to be non-compact and have cone points, one has that

$$\frac{h^2(M)}{4} \leq \lambda(M) \leq Ch(M)$$

where $\lambda(M) = \inf_f \frac{\int_M \|\nabla f\|^2\, dA}{\int_M f^2\, dA}$ with $\mathrm{Area}(\mathrm{supp}(f)) > 0$, see [7, Section 7]. In particular, if a sequence of surfaces M_n have $h(M_n)$ converging to 0 as n diverges to ∞, then also $\lambda_1(M_n)$ goes to 0, and vice-versa.

Due to its definition as an infimum, it is often possible to bound the Cheeger constant from above by finding a separating subset E, but computing the constant precisely is a more complicated endeavor. The author of [5] recently provided an algorithm which enables practical computation of a wide range of examples of hyperbolic surfaces. The examples presently of interest are (the orientation-preserving subgroups of) maximal arithmetic reflection groups; specifically, we study those which are not maximal Fuchsian groups. In [16] it was shown that eight of the ten such maximal reflection groups have the property that they are not *congruence* (see section 2 for definition).

Belolipetsky [4] asked about congruence in the context of obtaining a practical lower bound on λ_1 for maximal arithmetic reflection groups, because congruence groups have λ_1 bounded below by 975/4096 (Kim and Sarnak [15]), and conjecturally by 1/4 (Selberg [23]). We therefore seek to understand the geometry of those examples which are not congruence, with a view to establishing whether their values of λ_1 can be much smaller than these bounds. We prove the existence of small eigenvalues by searching for them and we compute the corresponding Maass cusp forms numerically. The smallest eigenvalue we find is approximately 0.14843, and in total we find that six of our non-congruence examples have λ_1 less than 1/4.

We note that all of the examples considered here have spectral gap at most 1/4, on account of the fact that the continuous spectrum of the Laplacian is $[1/4, \infty)$ [21] (see Section 4). We therefore compute the first discrete eigenvalue $\lambda_1^{\mathrm{disc}}$. We find two examples where the first discrete eigenvalue $\lambda_1^{\mathrm{disc}}(M)$ is much larger than $h(M)$. We find one further example where $\lambda_1^{\mathrm{disc}}(M) > h(M)$, but the values are still similar, and the remaining seven

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examples have $\lambda_1^{\text{disc}}(M) < h(M)$. This suggests that the three examples (one congruence, two non-congruence) with large λ_1^{disc} are distinguished geometrically from the other seven.

Since the examples in question are all index two subgroups of maximal arithmetic Fuchsian groups, which are necessarily all congruence, we also compute the Cheeger constant of the maximal Fuchsian group, with a view to studying how the Cheeger constant changes when passing to the subgroup. We find that the Cheeger constant seems to drop more dramatically when passing from a congruence group to a non-congruence subgroup than when passing to a congruence subgroup. More precisely, we find that in seven of the eight cases where the subgroup is non-congruence, h drops by more than half, and in the two cases where the subgroup is congruence, h drops by less than half.

This paper is organized as follows. In section 2, we provide some background and prove some helpful technical results. The algorithm that will be used to compute the Cheeger constant is described in section 3, and it is implemented in detail in sections 3.1–3.4. Section 3.5 summarizes our results on the Cheeger constant. In section 4, we compute and investigate Maass cusp forms. Our conclusions are stated in section 5.

2. Preliminaries

2.1. **Arithmetic reflection groups.** A cofinite hyperbolic reflection group is the group generated by reflections in the sides of a finite area hyperbolic polygon. Such a group is called *maximal* if it is not properly contained in another reflection group. The index two orientation-preserving subgroup of such a group will be referred to as its *rotation subgroup*; this is a Fuchsian group. A non-compact, cofinite, hyperbolic reflection group is *arithmetic* if it (and its rotation subgroup) is commensurable with the modular group $\mathrm{PSL}_2(\mathbb{Z})$; that is, if the reflection group has a finite index subgroup which is ($\mathrm{PSL}_2(\mathbb{R})$-conjugate to) a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Although there are infinitely many arithmetic reflection groups, it is known that there are only finitely many maximal arithmetic reflection groups: in dimension two this is due to Long, Maclachlan and Reid [18]; in dimension three to Agol [1]; and in other dimensions independently to Agol, Belolipetsky, Storm, and Whyte [2], and to Nikulin [20]. Note that this includes the result that arithmetic reflection groups do not exist above dimension 30.

For any positive integer n, there is a natural homomorphism

$$\psi_n : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$$

defined by reducing the matrix modulo n. The kernel of this homomorphism is a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ consisting of matrices congruent to $\pm\,\mathrm{Id}$; this subgroup is denoted $\Gamma(n)$ and is called the *principal congruence subgroup* of level n. A Fuchsian group commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ is called a *congruence* group if it contains some principal congruence subgroup.

It was shown by Helling [13, 14] that the maximal Fuchsian groups commensurable with the modular group are obtained by taking the normalizers $N(\Gamma_0(n))$ in $\mathrm{PSL}_2(\mathbb{R})$ of the groups

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \mid c \equiv 0 \mod n \right\}$$

for n a square-free integer. The modular group itself, as well as the twelve normalizers $N(\Gamma_0(n))$ for $n = 2, 3, 5, 6, 7, 10, 13, 14, 21, 30, 34, 39$ are rotation subgroups of reflection groups [16]. In these cases, the reflection groups are necessarily maximal arithmetic reflection groups, and all are necessarily congruence groups. The ten normalizers $N(\Gamma_0(n))$ for $n = 11, 15, 17, 19, 22, 26, 33, 42, 55, 66$ are not rotation subgroups of reflection groups, but each contains an index two subgroup which is. Again, these reflection groups are necessarily maximal arithmetic, but they are not necessarily congruence groups, and in fact, only the examples corresponding to $n = 15$ and $n = 17$ are congruence groups. The other eight examples are not congruence groups. These ten examples are the focus of the present study.

2.2. **Fuchsian groups and quotient surfaces.** In this section, we give some elementary results which will be useful in computing the Cheeger constants of our examples. We refer to Beardon [3] for background on the material in this section.

Lemma 2.1. *Suppose that Γ is a cofinite Fuchsian group and $\mathcal{F}$ is a convex, finite-sided fundamental domain for the action of Γ on $\mathbb{H}^2$. Suppose that there exists a finite collection of hyperbolic elements $\{\gamma_i\}_{i=1}^k \subset \Gamma$ such that:*

- *for each i, γ_i pairs two sides, s_i and s_i', of $\mathcal{F}$;*
- *for each i, $\mathrm{ax}(\gamma_i)$ intersects both s_i and s_i' orthogonally; and*
- *for each $i \neq j$, $\mathrm{ax}(\gamma_i) \cap \mathrm{ax}(\gamma_j) = \emptyset$.*

Suppose further that the axes $\mathrm{ax}(\gamma_i)$ *divide* $\mathcal{F}$ *into two regions, dark and light, such that each dark side is paired with another dark side, and each light side is paired with another light side. Then the images of the axes* $\mathrm{ax}(\gamma_i)$ *are geodesics on the quotient surface* $M = \Gamma\backslash\mathbb{H}^2$ *which together separate* M.

Proof. Since the axes are geodesics in $\mathbb{H}^2$, they project to geodesics on the quotient surface. Since $\mathrm{ax}(\gamma_i)$ intersects both s_i and s_i' orthogonally, gluing s_i to s_i' closes up the geodesic. The disjointness of the axes implies that the corresponding geodesics are disjoint on the quotient surface. Since like-shaded sides are identified with like, the only way to move from light side to dark is to cross an axis. $\square$

Lemma 2.2. *Let* α_1 *and* α_2 *be elliptic isometries of order 2 (henceforth called involutions) fixing* z_1 *and* $z_2 \neq z_1$ *respectively. Then the product* $\alpha = \alpha_1\alpha_2$ *is hyperbolic, the axis* $\mathrm{ax}\,(\alpha)$ *is the geodesic through* z_1 *and* z_2*, and the translation length of* α *along its axis is twice the distance between* z_1 *and* z_2.

This is found on page 174 of Beardon.

Lemma 2.3. *Let* Γ *be a non-cocompact, cofinite Fuchsian group with finite-sided fundamental domain* $\mathcal{F}$*, and let* $\ell > 0$*. Choose disjoint* Γ*-equivariant cusp horoball neighborhoods, and let* $\mathcal{F}'$ *denote (the closure of) the complement of these neighborhoods in* $\mathcal{F}$*. Let* $N(\mathcal{F}') = N_{\frac{\ell}{2}+\varepsilon}(\mathcal{F}')$ *denote the closed* $(\frac{\ell}{2}+\varepsilon)$*-neighborhood of* $\mathcal{F}'$*, and let* $G = \{\gamma_0 = \mathrm{Id}, \gamma_1, \ldots, \gamma_k\}$ *be a set of elements of* Γ *such that the union of translates* $\cup_{i=0}^{k}\gamma_i(\mathcal{F})$ *covers* $N(\mathcal{F}')$*. Then any geodesic of length at most* ℓ *on* $\Gamma\backslash\mathbb{H}^2$ *must correspond to an element* $\gamma_j\gamma_i^{-1}$ *for some* $0 \le i, j \le k$.

Proof. Let β be a closed geodesic on $\Gamma\backslash\mathbb{H}^2$ with length $d \le \ell$. Consider the lifts of β to the universal cover $\mathbb{H}^2$. Since β is not peripheral, it cannot lie entirely in a single cusp neighborhood, and since β is path-connected, it cannot lie entirely in the union of the disjoint cusp neighborhoods. Thus each lift of β has a point outside of the union of the equivariant horoball neighborhoods. There must be one lift which intersects $\mathcal{F}'$; let β' be such a lift, and choose a point $p_0 \in \beta' \cap \mathcal{F}'$.

The geodesic β' is the axis of a hyperbolic isometry $\gamma \in \Gamma$ with translation length d. There are two points on β', p_1 and p_2, each distance $\frac{d}{2}$ away from p_0, such that $\gamma(p_1) = p_2$. If either p_1 or p_2 is fixed by a finite order element of Γ, make a different choice of p_1 which is still on the axis of γ and within an $\frac{\varepsilon}{2}$-neighborhood of the inital choice, such that p_1 and $p_2 = \gamma(p_1)$ both have trivial stabilizers. Note that $p_1, p_2 \in N_{\frac{\ell}{2}+\varepsilon}(\mathcal{F}')$, and p_1 and p_2 are Γ-equivalent. As such, there exists a point $q \in \mathcal{F}$ and there exist group elements $\gamma_{i_1}, \gamma_{i_2} \in G$ such that $p_1 = \gamma_{i_1}(q)$ and $p_2 = \gamma_{i_2}(q)$. Since $q = \gamma_{i_1}^{-1}(p_1)$, it follows that $p_2 = \gamma_{i_2} \circ \gamma_{i_1}^{-1}(p_1)$. Since p_1 and p_2 were chosen to have trivial stabilizers, the only element of Γ which sends p_1 to p_2 is γ, and so $\gamma = \gamma_{i_2} \circ \gamma_{i_1}^{-1}$. Hence the original geodesic corresponds to an element $\gamma_j\gamma_i^{-1}$, for some $0 \le i, j \le k$, as required. $\square$

Lemma 2.4. *A finite coarea, genus zero Fuchsian group* Γ *with* t Γ*-inequivalent cusps, and* r *cone points of orders* $m_1, \ldots, m_r$ *respectively has coarea*

$$2\pi\left(-2 + t + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right).$$

This is stated as Theorem 10.4.3 of Beardon, with our t corresponding to s there, and the quantity referred to as t there always being 0 in our examples. In this paper, the genus g will always equal 0, and so the term $2(g-1)$ in the Theorem 10.4.3 formula becomes simply -2 for us.

3. The algorithm

In this section, we state the algorithm which is used to compute the Cheeger constant of our examples. This is an adapted version of the algorithm given in [5]; in particular, it has been simplified to reflect the fact that all the present examples are non-compact. We will keep track of two quantities: H will denote the current best estimate of the Cheeger constant h, and begins by taking the value $H = 1$ because M is assumed to have cusps and the isoperimetric ratio of a cusp neighborhood is 1; U represents the current upper bound on the total length of geodesics (or curves equidistant from a collection of geodesics) that could possibly result in a splitting which reduces H, and begins with the value $U = \mathrm{Area}(M)/2$.

(1) First, set $H = 1$ and $U = \mathrm{Area}(M)/2$.
(2) Select a collection $\{\gamma^{i_1}, \ldots, \gamma^{i_j}\}$ of geodesics which split M into two pieces A and B, and which have total length $\ell(\partial A) = \ell(\partial B)$ no greater than U.
(3) If $\mathrm{Area}(A) = \mathrm{Area}(B)$, then compute

$$H_0 = h^*(A) = \frac{\ell(\partial A)}{\mathrm{Area}(A)} = h^*(B) = \frac{\ell(\partial B)}{\mathrm{Area}(B)},$$

let $s = 0$, and proceed to Step 5.

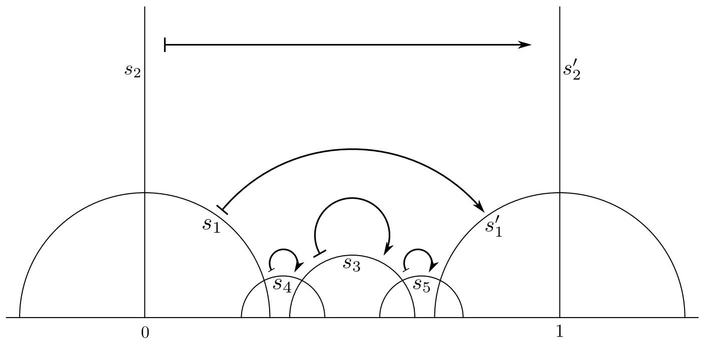

FIGURE 1. Ford domain $\mathcal{F}$ for $N(\Gamma_0(11))$.

(4) If $\mathrm{Area}(A) \neq \mathrm{Area}(B)$, without loss of generality let A denote the component with lesser area. Determine the minimum distance d_{i_j} perpendicular from the geodesics into B before the neighborhoods intersect, and minimize the maximum of

$$h^*(A_s) = \frac{\ell(\partial A)\cosh(s)}{\mathrm{Area}(A) + \ell(\partial A)\sinh(s)} \text{ and } h^*(B_s) = \frac{\ell(\partial B)\cosh(s)}{\mathrm{Area}(B) - \ell(\partial B)\sinh(s)}.$$

Let this minimum be H_0 and record the value of s for which this minimum occurs.

(5) If $H_0 < H$, then redefine $H = H_0$ and record the collection $(\{\gamma^{i_1}, \ldots, \gamma^{i_j}\}, s)$. If $H = H_0$, add the collection to the list of collections which achieve H. If $H_0 > H$, do nothing.

(6) If $H\,\mathrm{Area}(M)/2 < U$, redefine U as $H\,\mathrm{Area}(M)/2$; if not, leave U unchanged.

(7) Return to Step 2 until no further collections of geodesics satisfying the criterion in Step 2 can be found.

In practice, we will present a splitting by geodesics, and then argue why any other collection of geodesics which has length less than U does not result in a lesser value of H. As part of this, we will also check that no union of geodesic arcs with endpoints at cone points produces a lesser value of H.

To illustrate the general method, we show in detail the computation of the Cheeger constants of the surfaces M_{11} and N_{11}. We also show the computation of the Cheeger constant of M_{17} as one example where the Cheeger constant is realized by a curve equidistant from a geodesic, rather than by the geodesic itself.

3.1. **Example:** M_{11}**.** M_{11} is the quotient of $\mathbb{H}^2$ by the normalizer $N(\Gamma_0(11))$ in $\mathrm{PSL}_2(\mathbb{R})$ of the congruence subgroup $\Gamma_0(11)$. A Ford domain $\mathcal{F}$ for $N(\Gamma_0(11))$ is given below in Figure 1.

The domain $\mathcal{F}$ has area 2π, and the side-pairing elements which generate $N(\Gamma_0(11))$ are:

$$g_1 = \begin{pmatrix} \sqrt{11} & \frac{-1}{\sqrt{11}} \\ \sqrt{11} & 0 \end{pmatrix},\ g_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ g_3 = \begin{pmatrix} \sqrt{11} & \frac{-6}{\sqrt{11}} \\ 2\sqrt{11} & -\sqrt{11} \end{pmatrix},\ g_4 = \begin{pmatrix} \sqrt{11} & \frac{-4}{\sqrt{11}} \\ 3\sqrt{11} & -\sqrt{11} \end{pmatrix},\ g_5 = \begin{pmatrix} 2\sqrt{11} & \frac{-15}{\sqrt{11}} \\ 3\sqrt{11} & -2\sqrt{11} \end{pmatrix}.$$

The elements of $N(\Gamma_0(11))$ take two forms: those that belong to $\mathrm{PSL}_2(\mathbb{Z})$ have the form $\begin{pmatrix} a & b \\ 11c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$; those that are not have the form $\begin{pmatrix} x\sqrt{11} & \frac{y}{\sqrt{11}} \\ z\sqrt{11} & w\sqrt{11} \end{pmatrix}$, where $x, y, z, w \in \mathbb{Z}$. As such, the traces of elements in $N(\Gamma_0(11))$ are either integers, or integer multiples of $\sqrt{11}$.

To apply the algorithm, since M_{11} has cusps, we set $U = \pi$ and $H = 1$. We consider the geodesic corresponding to the hyperbolic element g_1. This element pairs the two sides s_1 and s_1'. Furthermore, the axis $\mathrm{ax}(g_1)$ meets both s_1 and s_1' orthogonally; this axis divides $\mathcal{F}$ into two pieces: the region A above the axis (dark) and B below (light); each dark side is paired with another, and each light side is paired with another (see Figure 2). Each of these two pieces has area π, and the length of the separating geodesic corresponding to g_1 is $2\,\mathrm{arccosh}\left(\frac{\sqrt{11}}{2}\right) \approx 2.18464$. Hence our $H_0 = h^*(A) = h^*(B) = \frac{2.18464}{\pi} \approx 0.695394$. This is less than $H = 1$, so this value becomes our new H, and we set $U = H\,\mathrm{Area}(M)/2 \approx 2.18464$.

Since in this case, the two pieces have equal area, the Cheeger constant can only be decreased if there exist simple closed curves, or a union of geodesic arcs between cone points, which separate M_{11} and which have total length less than U, and hence the corresponding elements have trace less than $\sqrt{11}$. We apply Lemma 2.3 with $\ell \approx 2.18464$,

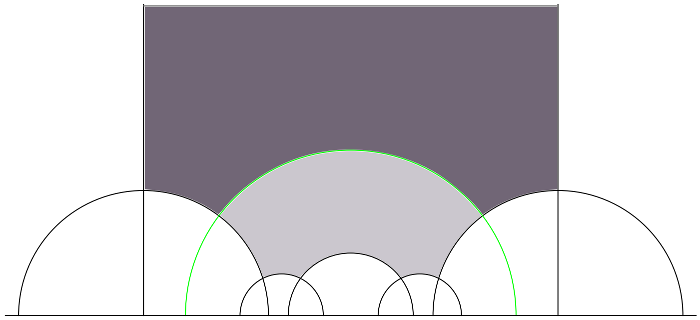

FIGURE 2. The axis of g_1 separates $\mathcal{F}$.

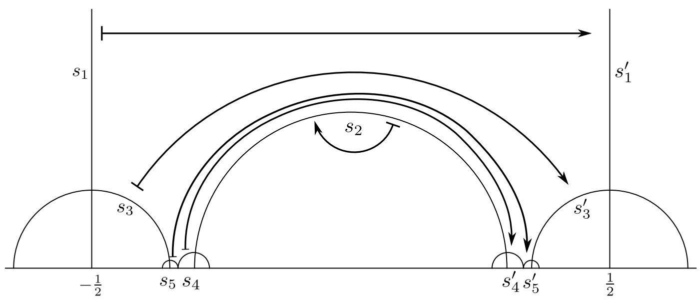

FIGURE 3. Ford domain for Γ'.

and we take $\frac{\ell}{2}+\varepsilon = 1.1$. We find that the set G of elements whose translates cover the 1.1-neighborhood of $\mathcal{F}$ is

$$G = \{\mathrm{Id}, g_1, g_2, g_3, g_4, g_5, g_1^{-1}, g_2^{-1}, g_2^{-1}g_1, g_2g_1^{-1}, g_1^{-1}g_5, g_4g_1^{-1}, g_5g_1, g_5g_3, g_1^{-1}g_3, g_1^{-1}g_3g_4, g_1g_3, g_1g_3g_5, g_3g_1, g_3g_1g_4, g_3g_1^{-1}, g_3g_1^{-1}g_5\}.$$

We search through elements of the form $\gamma_j\gamma_i^{-1}$ for $\gamma_i, \gamma_j \in G$ for hyperbolic elements with trace less than $\sqrt{11}$. We find that the only such elements are

$$\begin{pmatrix} 5 & -1 \\ 11 & -2 \end{pmatrix}, \begin{pmatrix} 9 & -5 \\ 11 & -6 \end{pmatrix},$$

and their inverses. Each of these is a product of two involutions: respectively, $g_3g_5g_3g_4$ and $g_5g_3g_4g_3$; and these elements are conjugate to one another by g_3. As such, there is exactly one geodesic arc on M_{11}, between two cone points, which corresponds to an element of trace 3, and hence, by Lemma 2.2, this arc has length arccosh $\left(\frac{3}{2}\right) \approx 0.962424$. The geodesic arc does not intersect itself and its endpoints are distinct cone points, so the arc does not separate M_{11}.

It remains to check whether there is another geodesic arc, between the same cone points, which together with this one separates M_{11}. Specifically, we seek geodesic arcs which are less than $2.18464 - 0.962424 = 1.22222$ in length. However, by considering 1.25-neighborhoods of each cone point, we see that there is no other such geodesic arc. Furthermore, these neighborhoods show that there are no two lifts of the same cone point which are distance less than U apart, and so there is no geodesic arc, with both endpoints at the same cone points, of length less than U. Hence we conclude that $h(M_{11}) \approx 0.695394$.

3.2. **Example:** N_{11}**.** The surface N_{11} is the quotient of $\mathbb{H}^2$ by the index two subgroup $\Gamma' \subset \Gamma$ generated by the following elements:

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 0 & \frac{-1}{\sqrt{11}} \\ \sqrt{11} & 0 \end{pmatrix}, \ \gamma_3 = \begin{pmatrix} \sqrt{11} & \frac{5}{\sqrt{11}} \\ 2\sqrt{11} & \sqrt{11} \end{pmatrix}, \ \gamma_4 = \begin{pmatrix} 10 & 3 \\ 33 & 10 \end{pmatrix}, \ \gamma_5 = \begin{pmatrix} 23 & 8 \\ 66 & 23 \end{pmatrix}.$$

A Ford domain, Q, for this group is given in Figure 3; for each i, γ_i identifies side s_i with s_i'. Note that these sides s_i are labeled differently than in the previous example.

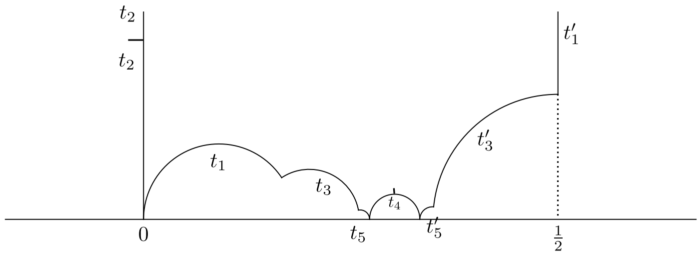

FIGURE 4. Alternative fundamental domain for Γ'.

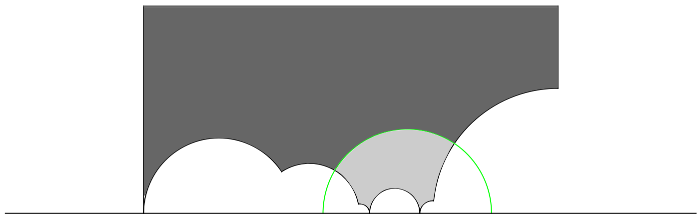

FIGURE 5. The axis of $\gamma_3\gamma_2$ separates N_{11}.

Since N_{11} has cusps, we set $U = 2\pi$ and $H = 1$. We first choose to study an element of Γ' of trace 3, in particular the element $\gamma_3\gamma_2$. Since this element does not identify sides of Q, we choose a different fundamental domain which has this element as a side-pairing. We do this by taking the left half of Q, i.e., the part of Q with negative real part (henceforth called Q^-), and we apply γ_2 to just Q^-. The resulting fundamental domain is shown in Figure 4.

In this fundamental domain, the side t_2 is paired with itself by γ_2; the sides t_1, t_3 and t_5 are identified with t_1', t_3' and t_5' respectively, by $\gamma_1\gamma_2$, $\gamma_3\gamma_2$ and $\gamma_5\gamma_2$ respectively, and the side t_4 is paired with itself by $\gamma_4\gamma_2$. The axis of $\gamma_3\gamma_2$ separates this fundamental domain into dark and light pieces (see Figure 5).

Since $\gamma_3\gamma_2$ has trace 3, and its axis separates, it corresponds to a geodesic of length $2\operatorname{arccosh}\left(\frac{3}{2}\right) \approx 1.92485$. This geodesic separates N_{11} into two pieces which each have area 2π. Our value of H_0 is therefore $\frac{1.92485}{2\pi} \approx 0.306349$, and is equal to $h^*(A)$ and $h^*(B)$. This value is less than one, and hence becomes our new H; we set $U = H\operatorname{Area}(M)/2 \approx 1.92485$.

In this case, because the element has trace 3, there cannot be a shorter geodesic simple closed curve on the surface, and because the two pieces each have area 2π, the minimum area is as large as possible. Therefore, the only way in which this value could not be the true value of $h(N_{11})$ would be if there were arcs joining cone points, of total length less than 1.92485, which separate N_{11}. But by Lemma 2.2, each pair of cone points are at least distance $\frac{1.92485}{2}$ apart. We must also account for the possibility that there is a geodesic arc with both endpoints at the same cone points. However, since there was no such arc of length less than 2.18 on M_{11}, there cannot be one on N_{11} of length less than the present U. Hence we conclude that $h(N_{11}) \approx 0.306349$.

3.3. **Other examples: the case $n = 17$.** To illustrate other facets of the algorithm, we now detail the computation of the Cheeger constant of $M_{17} = N(\Gamma_0(17))\backslash\mathbb{H}^2$ and of $N_{17} = \Gamma'\backslash\mathbb{H}^2$ for the index 2 subgroup Γ' of $N(\Gamma_0(17))$ which is the rotation subgroup of a maximal arithmetic reflection group.

Since 17 is prime, the elements of $N(\Gamma_0(17))$ take two forms: those that belong to $\mathrm{PSL}_2(\mathbb{Z})$ have the form $\begin{pmatrix} a & b \\ 17c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{Z}$; those that are not have the form $\begin{pmatrix} x\sqrt{17} & \frac{y}{\sqrt{17}} \\ z\sqrt{17} & w\sqrt{17} \end{pmatrix}$, where $x, y, z, w \in \mathbb{Z}$. As such, the traces of elements in $N(\Gamma_0(17))$ are either integers, or integer multiples of $\sqrt{17}$. By considering the congruence conditions on a and d from the restraint $ad - 17bc = 1$, and so $ad \equiv 1 \bmod 17$, we see that the least non-parabolic trace in $\Gamma_0(17)$ is 5. Hence the shortest geodesic has trace $\sqrt{17}$, and one such element is $\begin{pmatrix} \sqrt{17} & \frac{-1}{\sqrt{17}} \\ \sqrt{17} & 0 \end{pmatrix}$. This divides the surface into pieces A and B of areas π and 2π (see Figure 6), and since these areas are not equal, we proceed to extend A into B using closed curves equidistant from the geodesic.

We let s denote the distance extended into B. With reference to the equidistant curves in the Ford domain in Figure 6, we find that we may extend a distance of 1.28247 before the equidistant curve fails to be isotopic to the

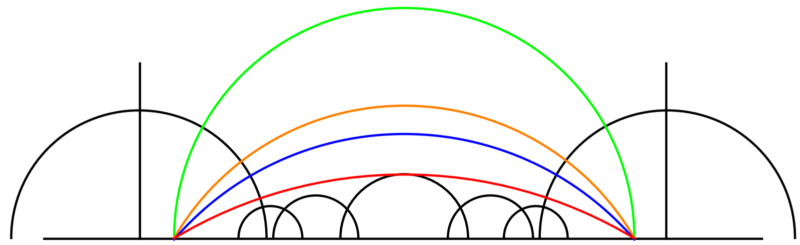

FIGURE 6. Equidistant curves from the geodesic.

geodesic. Using the equations

$$h^*(A_s) = \frac{\ell(\partial A)\cosh(s)}{\pi + \ell(\partial A)\sinh(s)} \text{ and } h^*(B_s) = \frac{\ell(\partial A)\cosh(s)}{2\pi - \ell(\partial A)\sinh(s)}$$

we find that $h^*(A_s)$ is minimized at $s = 0.779274$, and that $h^*(A_s) = h^*(B_s)$ when $s = 0.55247$. Thus $\max(h^*(A_s), h^*(B_s))$ is minimized at $s = 0.55247$, and here $H_0 = h^*(A_s) = h^*(B_s) = 0.663522$. Since this last number is less than $H = 1$, we set $H = 0.663522$ and $U = 3.12677$.

We next note that any other geodesic of trace $\sqrt{17}$ would produce a similar analysis. This is because the area formula (see Lemma 2.4) means that such a geodesic, if it separates, must divide the surface into subsurfaces of areas π and 2π respectively. Furthermore, as discussed above, the next smallest possible trace for a hyperbolic element in this group is 5, which corresponds to a geodesic of length 3.1336. This is larger than the present value of U, and so we need not consider any such elements.

It therefore remains to consider geodesic arcs between cone points. By considering neighborhoods of the fixed points of each involution and their translates by the group, we see that the minimum trace for a product of two involutions is 5. By Lemma 2.2, this means that the union of two geodesic arcs between distinct cone points must have length which exceeds U. It then remains to consider geodesic arcs which begin and end at the same cone point. We find one product of conjugate involutions with trace 19; by Lemma 2.2, this gives a geodesic arc of length 2.94166. This arc cuts off a subsurface of area less than π, and so the corresponding value of h^* is at least $2.94166/\pi \approx 0.936359$. All other geodesic arcs have length exceeding U. We therefore conclude that $h(M_{17}) \approx 0.663522$.

The surface N_{17} is the quotient of $\mathbb{H}^2$ by an index two subgroup of $N(\Gamma_0(17))$. This surface has area 6π, so we begin by setting $H = 1$ and $U = 3\pi$. We find a geodesic of trace 6 which divides it into two equal pieces, each of area 3π. Here, $H_0 = h^*(A) = h^*(B) \approx 0.374067$; this value becomes our new H, and we set $U \approx 3.52549$. We find no shorter geodesics than this, and we also find no combinations of arcs between cone points of total length less than U. Thus $h(N_{17}) \approx 0.374067$.

3.4. **Other examples: the case $n = 33$.** Since it exhibits different characteristics from the other values of n, we will describe the results of the algorithm for the case of $n = 33$. Before doing this, we first describe how it differs from the other examples.

In all the cases we consider, the Cheeger constant of the minimal surface M_n, corresponding to the maximal Fuchsian group $N(\Gamma_0(n))$, is realized by either a closed geodesic (as in the case $n = 11$) or an equidistant neighborhood (as in the case $n = 17$). In the other nine cases where $n \neq 33$, the Cheeger constant of N_n is realized as a geodesic which passes through two of the cone points of M_n which do not lift (i.e., the involutions are not found in the index 2 subgroup). In the case $n = 33$, these cone points are a little farther apart, and so the Cheeger constant of N_{33} is realized by a different closed geodesic.

In the case of M_{33}, we find that the Cheeger constant is $h(M_{33}) \approx 0.740622$ and this is realized by an equidistant curve to a geodesic of trace $\sqrt{33}$. We omit the details of this computation as they are similar to those of M_{17}.

To find $h(N_{33})$, we begin by taking the geodesic described above. It has trace 14, and divides the surface into two pieces, each of area 4π. The corresponding Cheeger estimate is $H \approx 0.419201$, and $U \approx 5.26783$. We now apply Lemma 2.3 with $\frac{\ell}{2} + \varepsilon = 2.64$, and search for elements with trace less than 14. Up to conjugation, we find six geodesics, with traces $\sqrt{33}$, $4\sqrt{3}$, $2\sqrt{33}$, $7\sqrt{3}$, 13, and $8\sqrt{3}$ respectively. In all cases but trace 13, the geodesic does not divide the surface into pieces of equal area, and hence the corresponding H_0 is larger than H. In the case of the geodesic of trace 13, the geodesic divides the surface into two pieces of equal area 4π, and so the corresponding $H_0 \approx 0.407274$ (see Figure 7). Since we already enumerated all of the closed geodesics of trace less than 14, it

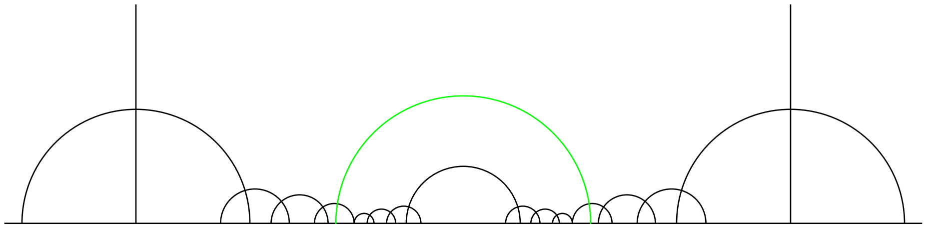

Figure 7. A trace 13 geodesic splits N_{33} into two pieces of area 4π.

Table 1. Cheeger constants. The asterisk * denotes cases where N_n is a congruence surface.

n	$h(M_n)$	$h(N_n)$
11	0.695394	0.306349
15*	0.814548	0.498728
17*	0.663522	0.374067
19	0.672365	0.183809
22	0.717333	0.279467
26	0.719348	0.239543
33	0.740622	0.407274
42	0.772596	0.328406
55	0.485381	0.204233
66	0.472607	0.218937

remains to check that there are no combinations of arcs between cone points with total length less than $U \approx 5.11796$. There are no such arcs, so we conclude that $h(N_{33}) \approx 0.407274$.

3.5. **Summary of results.** Table 1 lists numerical values of the Cheeger constants of a collection of hyperbolic surfaces M_n and N_n. These values of the Cheeger constants result from the following considerations:

- $\underline{n = 11}$: M_{11} has area 2π and a geodesic with trace $\sqrt{11}$ which divides it into two equal pieces; thus $h(M_{11}) \approx 0.695394$. N_{11} has area 4π and a geodesic with trace 3 which divides it into two equal pieces; thus $h(N_{11}) \approx 0.306349$.
- $\underline{n = 15}$: M_{15} has area 2π and a geodesic with trace $\sqrt{15}$ which divides it into two equal pieces; thus $h(M_{15}) \approx 0.814548$. N_{15} has area 4π and a geodesic with trace 5 which divides it into two equal pieces; thus $h(N_{15}) \approx 0.498728$.
- $\underline{n = 17}$: M_{17} has area 3π and a geodesic with trace $\sqrt{17}$ which divides it into two pieces of areas π and 2π; an equidistant curve from this at distance $s = 0.55247$ gives $h(M_{17}) \approx 0.663522$. N_{17} has area 6π and a geodesic with trace 6 which divides it into two equal pieces; thus $h(N_{17}) \approx 0.374067$.
- $\underline{n = 19}$: M_{19} has area $10\pi/3$ and a geodesic with trace $\sqrt{19}$ which divides it into two pieces of areas π and $7\pi/3$; an equidistant curve from this at distance $s = 0.685239$ gives $h(M_{19}) \approx 0.672365$. N_{19} has area $20\pi/3$ and a geodesic with trace 3 which divides it into equal pieces; thus $h(N_{19}) \approx 0.183809$.
- $\underline{n = 22}$: M_{22} has area 3π and a geodesic with trace $\sqrt{22}$ dividing it into two pieces of areas π and 2π; an equidistant curve from this at distance $s = 0.503269$ gives $h(M_{22}) \approx 0.717333$. N_{22} has area 6π and a geodesic with trace 4 which divides it into equal pieces; thus $h(N_{22}) \approx 0.279467$.
- $\underline{n = 26}$: M_{26} has area $7\pi/2$ and a geodesic with trace $\sqrt{26}$ dividing it into two pieces of areas π and $5\pi/2$; an equidistant curve from this at distance $s = 0.686576$ gives $h(M_{26}) \approx 0.719348$. N_{26} has area 7π and a geodesic with trace 4 which divides it into equal pieces; thus $h(N_{26}) \approx 0.239543$.
- $\underline{n = 33}$: M_{33} has area 4π and a geodesic with trace $\sqrt{33}$ dividing it into two pieces of areas π and 3π; an equidistant curve from this at distance $s = 0.820071$ gives $h(M_{33}) \approx 0.740622$. N_{33} has area 8π and a geodesic with trace 13 which divides it into equal pieces; thus $h(N_{33}) \approx 0.407274$.
- $\underline{n = 42}$: M_{42} has area 4π and a geodesic with trace $\sqrt{42}$ dividing it into two pieces of areas π and 3π; an equidistant curve from this at distance $s = 0.772596$ gives $h(M_{42}) \approx 0.771086$. N_{42} has area 8π and a geodesic with trace 8 which divides it into equal pieces; thus $h(N_{42}) \approx 0.328406$.

- $\underline{n = 55:}$ M_{55} has area 6π and a geodesic with trace $3\sqrt{11}$ dividing it into two equal pieces of areas 3π and 3π; this gives $h(M_{55}) \approx 0.485381$. N_{55} has area 12π and a geodesic with trace 7 which divides it into equal pieces; thus $h(N_{55}) \approx 0.204233$.
- $\underline{n = 66:}$ M_{66} has area 6π and a geodesic with trace $2\sqrt{22}$ dividing it into two equal pieces of areas 3π and 3π; this gives $h(M_{66}) \approx 0.472607$. N_{66} has area 12π and a geodesic with trace 8 which divides it into equal pieces; thus $h(N_{66}) \approx 0.218937$.

We note that the upper bounds for the Cheeger constants of N_{22} and N_{26} which appear in [16, Table 4.2] are erroneous, as they are underestimates of the true Cheeger constants.

4. Maass cusp forms

In this section, we compute Maass cusp forms on a collection of hyperbolic surfaces M_n and N_n for various values of n.

Let $M = \Gamma\backslash\mathbb{H}^2$ be the quotient surface of a cofinite, but non-cocompact Fuchsian group $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$, and let $\chi : \Gamma \to \mathbb{C} \setminus \{0\}$ be a multiplicative character.

A *Maass form* on (Γ, χ) is a

real analytic, $f \in C^\infty(\mathbb{H}^2)$,
square-integrable, $f \in L^2(M)$,
automorphic, $\chi(\gamma)f(\gamma z) = f(z)\ \forall\gamma \in \Gamma$,
eigenfunction of the Laplace-Beltrami operator, $-\Delta f(z) = \lambda f(z)$.

If a Maass form on (Γ, χ) vanishes in all the cusps of M, it is called a *Maass cusp form.*

The Laplace-Beltrami operator is an essentially self-adjoint operator on M, hence its eigenvalues are real. The spectral resolution on M consists of three parts, the zero eigenvalue $\lambda_0 = 0$, spanned by the constant eigenfunction, the continuous spectrum $\lambda \in [1/4, \infty)$, spanned by certain Maass forms, the so called Eisenstein series, and a countable set of discrete eigenvalues $0 < \lambda_1 \le \lambda_2 \le \ldots$ which are spanned by Maass cusp forms [21, 22].

We are particularly interested in the first discrete eigenvalue λ_1 which we compute numerically using Hejhal's algorithm [11] together with strategies for finding eigenvalues [24, end of §2 and §3] and [25].

We compute on arithmetic reflection groups and on finite index subgroups thereof. If Γ' is a finite index rotation or reflection subgroup of Γ, then Γ' is invariant with respect to conjugation by $\gamma \in \Gamma$. The surface $M' = \Gamma'\backslash\mathbb{H}^2$ has symmetries and can be tesselated by non-overlapping copies of the surface $M = \Gamma\backslash\mathbb{H}^2$,

$$M' = \cup_{\Gamma/\Gamma'}\gamma M.$$

The Laplace-Beltrami operator is invariant with respect to isometries, and in particular to the operations of the symmetry group Γ/Γ'. Maass forms on (Γ', Id) fall into symmetry classes, where the symmetry classes are represented by multiplicative characters χ,

$$\chi(\gamma)f(\gamma z) = f(z) \quad \forall\gamma \in \Gamma.$$

By automorphy on (Γ', Id) we have

$$f(\gamma z) = f(z) \quad \forall\gamma \in \Gamma'.$$

Consequently, a Maass form on (Γ', Id) is a Maass form on (Γ, χ), where χ is a multiplicative character on Γ with $\chi(\gamma) = 1\ \forall\gamma \in \Gamma'$. The symmetry of the Maass form is specified by the values of χ on Γ/Γ'. Since every symmetry operation $\gamma \in \Gamma/\Gamma'$ has finite order, the multiplicative character takes values that are roots of unity.

4.1. **Example:** M_{19} **and** N_{19}.

The index two subgroup Γ' of $\Gamma = N(\Gamma_0(19))$ is generated by

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \gamma_3 = \begin{pmatrix} \sqrt{19} & \frac{9}{\sqrt{19}} \\ 2\sqrt{19} & \sqrt{19} \end{pmatrix}, \gamma_4 = \begin{pmatrix} \sqrt{19} & \frac{6}{\sqrt{19}} \\ 3\sqrt{19} & \sqrt{19} \end{pmatrix}, \gamma_7\gamma_5\gamma_7\gamma_5 = \begin{pmatrix} 39 & 10 \\ 152 & 39 \end{pmatrix},$$

$$\gamma_7\gamma_5\gamma_7\gamma_1\gamma_5 = \begin{pmatrix} 37 & 9 \\ 152 & 37 \end{pmatrix}, \gamma_7\gamma_5\gamma_7\gamma_3\gamma_5 = \begin{pmatrix} 7\sqrt{19} & \frac{31}{\sqrt{19}} \\ 30\sqrt{19} & 7\sqrt{19} \end{pmatrix}, \gamma_7\gamma_5\gamma_7\gamma_4\gamma_5 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{19}} \\ \sqrt{19} & 0 \end{pmatrix},$$

and Γ' is invariant with respect to conjugation by the involution and the reflection,

$$\gamma_5 = \begin{pmatrix} -\sqrt{19} & -\frac{5}{\sqrt{19}} \\ 4\sqrt{19} & \sqrt{19} \end{pmatrix} \quad \text{and} \quad \gamma_7 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

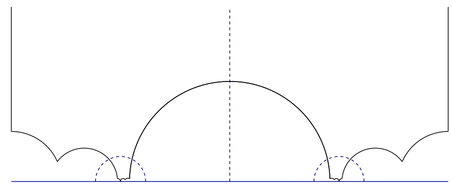

FIGURE 8. A fundamental domain for N_{19}. N_{19} is symmetric with respect to the involution γ_5 and the reflection γ_7. The symmetries are indicated as dotted lines.

The conjugation relations of the generators are

$$\begin{aligned}
\gamma_5^{-1}\gamma_1\gamma_5 &= (\gamma_7\gamma_5\gamma_7\gamma_5)^{-1}(\gamma_7\gamma_5\gamma_7\gamma_1\gamma_5), & \gamma_7^{-1}\gamma_1\gamma_7 &= \gamma_1^{-1},\\
\gamma_5^{-1}\gamma_3\gamma_5 &= (\gamma_7\gamma_5\gamma_7\gamma_5)^{-1}(\gamma_7\gamma_5\gamma_7\gamma_3\gamma_5), & \gamma_7^{-1}\gamma_3\gamma_7 &= \gamma_3^{-1},\\
\gamma_5^{-1}\gamma_4\gamma_5 &= (\gamma_7\gamma_5\gamma_7\gamma_5)^{-1}(\gamma_7\gamma_5\gamma_7\gamma_4\gamma_5), & \gamma_7^{-1}\gamma_4\gamma_7 &= \gamma_4^{-1},\\
\gamma_5^{-1}(\gamma_7\gamma_5\gamma_7\quad\gamma_5)\gamma_5 &= (\gamma_7\gamma_5\gamma_7\gamma_5)^{-1}\quad, & \gamma_7^{-1}(\gamma_7\gamma_5\gamma_7\quad\gamma_5)\gamma_7 &= (\gamma_7\gamma_5\gamma_7\quad\gamma_5)^{-1},\\
\gamma_5^{-1}(\gamma_7\gamma_5\gamma_7\gamma_1\gamma_5)\gamma_5 &= (\gamma_7\gamma_5\gamma_7\gamma_5)^{-1}\gamma_1, & \gamma_7^{-1}(\gamma_7\gamma_5\gamma_7\gamma_1\gamma_5)\gamma_7 &= (\gamma_7\gamma_5\gamma_7\gamma_1\gamma_5)^{-1},\\
\gamma_5^{-1}(\gamma_7\gamma_5\gamma_7\gamma_3\gamma_5)\gamma_5 &= (\gamma_7\gamma_5\gamma_7\gamma_5)^{-1}\gamma_3, & \gamma_7^{-1}(\gamma_7\gamma_5\gamma_7\gamma_3\gamma_5)\gamma_7 &= (\gamma_7\gamma_5\gamma_7\gamma_3\gamma_5)^{-1},\\
\gamma_5^{-1}(\gamma_7\gamma_5\gamma_7\gamma_4\gamma_5)\gamma_5 &= (\gamma_7\gamma_5\gamma_7\gamma_5)^{-1}\gamma_4, & \gamma_7^{-1}(\gamma_7\gamma_5\gamma_7\gamma_4\gamma_5)\gamma_7 &= (\gamma_7\gamma_5\gamma_7\gamma_4\gamma_5)^{-1}.
\end{aligned}$$

The surface $N_{19} = \Gamma'\backslash\mathbb{H}^2$ has two inequivalent cusps. A fundamental domain for N_{19} is displayed in Figure 8.

Maass forms on (Γ', Id) fall into four symmetry classes,

(++) $f(\gamma_7 z) = f(z)$, $f(\gamma_5 z) = f(z)$,
(+-) $f(\gamma_7 z) = f(z)$, $f(\gamma_5 z) = -f(z)$,
(-+) $f(\gamma_7 z) = -f(z)$, $f(\gamma_5 z) = f(z)$,
(--) $f(\gamma_7 z) = -f(z)$, $f(\gamma_5 z) = -f(z)$.

For ease of notation, we will refer to these four symmetry classes as (++), (+-), (-+), and (--), respectively.

Let $\Gamma = N(\Gamma_0(19))$ be the orientation preserving arithmetic maximal reflection group and let $\hat\Gamma = \langle \Gamma, \gamma_7\rangle$ be the non-orientation preserving arithmetic maximal reflection group. Moreover, let $M_{19} = \Gamma\backslash\mathbb{H}^2$ and $\hat M_{19} = \hat\Gamma\backslash\mathbb{H}^2$ be the corresponding quotient surfaces. Maass forms on (Γ', Id) with symmetry χ are Maass forms on $(\hat\Gamma, \chi)$ and vice-versa.

Computing Maass cusp forms on $(\hat\Gamma, \chi)$, we need to take care of one cusp only, while we obtain the discrete spectra on N_{19} and M_{19},

$$\begin{aligned}
\mathrm{spec}_{\mathrm{discr}}(N_{19}) &= \mathrm{spec}_{\mathrm{discr}}(\hat\Gamma, \texttt{(++)}) \cup \mathrm{spec}_{\mathrm{discr}}(\hat\Gamma, \texttt{(+-)}) \cup \mathrm{spec}_{\mathrm{discr}}(\hat\Gamma, \texttt{(-+)}) \cup \mathrm{spec}_{\mathrm{discr}}(\hat\Gamma, \texttt{(--)}),\\
\mathrm{spec}_{\mathrm{discr}}(M_{19}) &= \mathrm{spec}_{\mathrm{discr}}(\hat\Gamma, \texttt{(++)}) \cup \mathrm{spec}_{\mathrm{discr}}(\hat\Gamma, \texttt{(-+)}).
\end{aligned}$$

The first few eigenvalues of $\mathrm{spec}_{\mathrm{discr}}(\hat\Gamma, \chi)$, and hence, for N_{19} and M_{19} are listed in Table 2.

4.2. **Maass cusp forms on arithmetic maximal reflection groups with two ideal vertices.** There are ten non-cocompact arithmetic maximal reflection groups with two ideal vertices which we consider [16, §4.3]. These are the groups of level $11, 15, 17, 19, 22, 26, 33, 42, 55, 66$ on which we compute Maass cusp forms numerically.

Similarly as before, Maass forms on these groups fall into four symmetry classes. For each of the symmetry classes (++), (-+), and (--), there are infinitely many Maass cusp forms per level. For the symmetry class (+-) there are also infinitely many Maass cusp forms for level 15 and 17, but there is at most one Maass cusp form per level $11, 19, 22, 26, 33, 42, 55, 66$. The reason is that for the latter levels the group is not congruence.

We are interested in small eigenvalues. The number of Maass cusp forms with eigenvalue less than 200 is listed in Table 3 in dependence of the level and the symmetry class. The smallest eigenvalue in dependence of the level and the symmetry class is listed in Table 4. The upper bounds of [16, Table 4.2] are respected.

The first discrete eigenvalue on N_{17} is elusive. It belongs to a CM form.

4.3. **CM forms.** CM forms are Maass cusp forms that arise as theta-lifts from GL(1). CM forms with eigenvalue $\lambda = 1/4$ were first constructed by Hecke [10]. Later, Maass gave a more general construction of CM forms and identified them with Maass cusp forms [19].

CM forms live on congruence subgroups of certain levels. On arithmetic maximal reflection groups with two ideal vertices, there are CM forms for level $n = 17$ (only). By construction, they are in symmetry class (+-).

TABLE 2. Consecutive eigenvalues λ of Maass cusp forms on the index two subgroup Γ' of $\Gamma = N(\Gamma_0(19))$ in dependence of the symmetry class. We note that there is only one eigenvalue for symmetry class (+-).

λ symmetry clsss (++)	λ symmetry class (+-)	λ symmetry class (-+)	λ symmetry class (--)
2.018365089	0.169612040	5.526238214	5.623644096
5.178444802		10.44255994	11.09994133
8.183549674		11.98729193	13.34726954
9.893410725		17.09642816	19.16508797
16.96819424		19.89417157	21.71327432
18.55690731		23.09408224	24.65978283
18.59544749		27.31732822	28.16059477
21.32460096		30.03293145	30.44847115
24.00366338		31.23572386	36.66774935
30.96013885		38.37766641	37.67237085
32.92132204		40.77536109	40.74040666
33.39522368		41.99782317	43.19669501
36.12307204		45.98970332	46.18071110
41.54143328		49.35948927	49.21126748
41.84058775		49.68599715	52.24479435
44.25260852		55.81902339	54.63308313
46.79262854		58.65413871	57.51914857
51.16589334		60.25006194	61.12735841
54.93920130		63.93528420	63.81506781
$\vdots$		$\vdots$	$\vdots$

TABLE 3. The number of Maass cusp forms with eigenvalue less than 200 in dependence of the level and the symmetry class.

level	$\#\{\lambda \in (0, 200]\}$ symmetry class (++)	$\#\{\lambda \in (0, 200]\}$ symmetry class (+-)	$\#\{\lambda \in (0, 200]\}$ symmetry class (-+)	$\#\{\lambda \in (0, 200]\}$ symmetry class (--)
11	37	1	38	38
15	36	37	38	38
17	63	58	63	60
19	71	1	68	68
22	60	1	61	61
26	74	1	72	72
33	86	0	84	84
42	85	0	83	83
55	140	1	128	127
66	138	1	129	128

Consider the level $n = 17$. Let $F = \mathbb{Q}(\sqrt{17})$ be a real quadratic field. Let $\eta = 4 + \sqrt{17}$ be the usual fundamental unit for $\mathcal{O}_F$. Let k be an arbitrary integer and $r = \frac{\pi k}{\log \eta}$ be the spectral parameter. Then, $\lambda = r^2 + 1/4$ is the eigenvalue of a CM form on the surface N_{17}. If $k \neq 0$, the Fourier expansion coefficients of these CM forms are explicitely given in [12],

The CM form for $k = 0$ is special and needs to be treated with care. According to the definition of Maass cusp forms, they should be square-integrable. Obviously, this is the case for all Maass cusp forms (CM forms included), except for the CM form with eigenvalue $\lambda = 1/4$. The latter has the term $a_0 y^{1/2}$ in its Fourier expansion whose L^2-norm diverges logarithmically.

The CM form with eigenvalue $\lambda = 1/4$ is elusive. Its L^2-norm exists in the sense of a distribution and its Fourier coefficients read

$$a_0 = \log \eta, \quad a_1 = 1, \quad a_p = \left(\frac{n}{p}\right)_{\mathrm{Kr}} + 1 \quad \text{for } p \text{ prime}, \quad a_{n_1} a_{n_2} = \sum_{l | (n_1, n_2)} \left(\frac{n}{l}\right)_{\mathrm{Kr}} a_{n_1 n_2 / l^2},$$

Table 4. The smallest eigenvalue on non-cocompact arithmetic maximal reflection groups in dependence of the level and the symmetry class. On the non-congruence subgroups of level 33 and 44 there are no Maass cusp forms of symmetry class (+-).

level	λ symmetry class (++)	λ symmetry class (+-)	λ symmetry class (-+)	λ symmetry class (--)
11	6.41822455110	0.24456267323	9.06024545639	10.00253800339
15	10.62011651411	3.56777601683	5.82589930819	9.42106297240
17	3.67134534972	0.25000000000	4.12297031172	6.24399453828
19	2.01836508907	0.16961204041	5.52623821363	5.62364409573
22	3.24185004329	0.23828308162	4.93732929346	5.81410275753
26	2.14598046442	0.21588714472	4.46353001031	5.05501819375
33	2.24813673858		2.67134067803	4.83385807583
42	1.87318457587		2.92045953841	4.43972502521
55	0.68768162820	0.14843098682	2.76461871050	3.30378314566
66	0.62303205749	0.15286533999	2.67134067803	3.43082014307

where $\left(\frac{n}{l}\right)_{\mathrm{Kr}}$ is the Kronecker symbol, a multiplicative character modulo the level n.

5. Conclusions

The most immediate conclusion to be drawn from the results above is that there exist maximal arithmetic hyperbolic reflection groups with $\lambda_1 < 1/4$. Further, the first eigenvalue for these groups can be lower than it must be for congruence groups, and so one cannot hope that there is a lower bound for λ_1 corresponding with the known lower bounds for congruence groups. The smallest first eigenvalue found here is 0.14843, but we have no reason to believe that this should itself serve as a lower bound for λ_1 amongst all maximal arithmetic hyperbolic reflection groups.

We note that for all of the examples considered here, the Cheeger constant $h \leq 1$, which is true because these are cusped hyperbolic surfaces, and a cusp neighborhood always has Cheeger ratio equal to 1. In all of the cases here where $\lambda_1 \leq 1$ on N_n, we find that $\lambda_1 < h$, so the Cheeger constant serves as an upper bound for the first eigenvalue. Further investigation may be helpful here in order to understand whether, and in what circumstances, we may take the Cheeger constant to serve as an upper bound for λ_1, and in which circumstances we have $\lambda_1 < 1$.

It is interesting to compare each Cheeger constant $h(M_n)$ with that of its double cover $h(N_n)$. Recall that each surface M_n is congruence. With the exception of the case $n = 33$, when the cover N_n is not congruence, the ratio $\dfrac{h(N_n)}{h(M_n)}$ is less than 1/2. In both cases where N_n is congruence, the ratio is larger than 1/2. This raises the following question.

Question. Suppose Γ is a congruence subgroup commensurable with $\mathrm{PSL}_2(\mathbb{Z})$, $\Gamma' < \Gamma$ an index two subgroup, and that $h(\mathbb{H}^2/\Gamma')/h(\mathbb{H}^2/\Gamma) < 1/2$. Must Γ' then be a non-congruence group?

It is perhaps interesting to note that the cases $n = 15$ and $n = 42$ produce examples of orbifold surfaces N_{15} and N_{42} where $\lambda_1^{\mathrm{disc}}$ is so large that using it instead of the spectral gap (in this case 1/4) would result in examples which appear to fail to satisfy Buser's inequality. Specifically, for N_{15}, we have $h = 0.498728$, and $\lambda_1^{\mathrm{disc}} = 3.5678$, which exceeds $2h + 10h^2 = 3.4848$. For N_{42} we have $h = 0.328406$ and $\lambda_1^{\mathrm{disc}} = 1.87318$, which exceeds $2h + 10h^2 = 1.7353$. Recall that Buser's inequality is formulated for closed manifold surfaces, and our examples have cusps and cone points. Nevertheless, these examples are worthy of further investigation, as this observation may indicate the presence of some different geometric properties which correspond to higher discrete eigenvalues.

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