About perspectives of nonstandard analysis

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Nonstandard analysis (NSA) is developing in two directions. The most part of researches are applying nonstandard analysis in some particular areas of classical mathematics. They consider NSA as a powerful method of discovering and proving new results or of simplifying and making more intuitively clear complicated proofs of known theorems. The second part of researches studies the structures of NSA by themselves as interesting from their point of view mathematical objects. They introduce and study various axiomatizations of NSA, develop descriptive set theory over internal sets, and study some standard objects constructed from nonstandard ones. I mean first of all the Loeb measures and nonstandard hulls of Banach spaces that are most often used in applications. So, the intersection of these two parts of researchers is non-empty. However, the second part is much smaller, than the first one.

Since the emergence of nonstandard analysis in the middle of 60th of the last century, a lot of results were obtained by the first group in various areas of mathematics from mathematical physics to number theory. Among the results of the last decade that attracted the attention of mathematical community the work of R. Jin in applications of nonstandard analysis to additive number theory [**?**] and the work of I. Goldbring that contains the solution of the 5th Hilbert problem for local groups [**?**] should be mentioned.

Though the mathematicians, who work in standard mathematics based on Cantor's Set Theory (let us call them for shortness classical mathematicians) are often interested in results obtained by nonstandard methods, they are usually reluctant to accept the proofs based on NSA. It is a typical situation, when a paper containing some results obtained by using NSA is followed by a paper that contains the standard proofs of these or even more general results. It may happen that these standard proofs are based on some absolutely other ideas, than the proofs based on NSA. I think that the reason of this reluctance is that NSA is the overwhelming formalization of mathematics, that is natural and necessary in mathematical logic, but irrelevant or most of the other areas of mathematics. The transfer principle of nonstandard analysis that is the main tool of obtaining classical results by nonstandard methods is a very formal logic theorem (Los theorem) applied to the very formal system (structure over $\mathbb{R}$). It is rather hard to accept this theorem for a non-logicians, who prefer to invent their own proofs or to modify the nonstandard proofs in "normal" language.

Practically all applications of NSA to the classical mathematics are based on the models of NSA that are ultrapowers of $\mathbb{R}$ and most of the NSA proofs of classical results can be rewritten in terms of ultraproducts and limits along ultrafilters with a minor loss of simplicity. These proofs are more acceptable for the classical mathematicians. In some recent papers (see e.g. [**?**, **?**]) the exposition is based directly on ultrapowers and ultralimits without even mentioning of NSA. Indeed, the ultraproducts were used in the theories of Banach spaces and von Neumann algebras far before the creation of NSA. However, for many of applications the language of NSA provides some good intuition for finding proofs and even for finding the formulations of theorems. For example, many applications of nonstandard analysis (NSA) to the investigation of infinite structures are based on embedding of these structures in appropriate hyperfinite structures. The latter inherit many properties of finite structures due to the transfer principle of NSA. The effectiveness of the described applications of NSA is based on the fact that many properties can be proved in a more straightforward way for finite structures, than for the corresponding infinite ones.

In the framework of standard mathematics the same idea is implemented by an approximation of an infinite structure by finite ones, e.g. by an embedding of this infinite structure into an appropriate inductive (projective) limit of finite structures. However, in some problems a construction of inductive and projective limits is either impossible or inappropriate, while the hyperfinite structures and their nonstandard hulls can be effective. One can find many such examples in stochastic analysis, functional analysis, harmonic analysis and mathematical physics (see e.g. the books [**?**, **?**, **?**, **?**] and the bibliography therein).

It seems to me that the results in this direction of NSA will appear from time to time with more or less the same degree of intensity as in previous decades. However, it is hard to expect that this direction in NSA will confirm the well known prediction of Kurt Goedel: "...there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future."

In my opinion, the second direction in NSA has more chances to justify this prediction. The reason for this opinion is based on the extremely important role of computers in modern investigations in science and mathematics. One of consequences of this role is the new point of view on the interaction between the discrete and continuous mathematics that is shared by more and more especially applied mathematicians. The best of all this role was expressed by D. Zeilberger in [**?**]: Continuous analysis and geometry are just degenerate approximations to the discrete world... While discrete analysis is conceptually simpler ... than continuous analysis, technically it is usually much more difficult. Granted, real geometry and analysis were necessary simplifications to enable humans to make progress in science and mathematics..... So, according to this point of view the continuous mathematics is an approximation of the discrete one in contraposition to the traditional point of view.

Under this approach the notion of a very big finite set is very important and the definition of a hyperfinite set in NSA is an appropriate formalization of this notion that satisfies the modern requirements to mathematical rigor. The notion of a "very big quantity" cannot be formalized in the framework of the Cantor Set Theory. It depends on a problem under consideration. However, NSA provides some rigorous logical rules that allow to make arguments with such notions and prove some qualitative results involving them.

Some examples of such statements were introduced in the paper [**?**]. In this paper the behavior of ergodic means of discrete time dynamical systems on a very big finite probability space Y is discussed. The G. Birkhoff Ergodic Theorem states the eventual stabilization of ergodic means of integrable functions for almost all points of the probability space. The trivial proof of this theorem for the case of finite probability spaces shows that this stabilization happens for those time intervals, whose length n exceeds significantly the cardinality $|Y|$ of Y, i.e. $n/|Y|$ is a very big number. For the case of a very big number $|Y|$ we consider the class of S-integrable functions on Y . The class of S-integrable functions is an analog of the class of integrable functions on an infinite probability space. It is shown that the behavior of ergodic means of S-integrable functions demonstrates some regularity even for those intervals, whose length is comparable with $|Y|$. The ergodic means A_n and A_m on the intervals of time $\{0, ..., m-1\} = \widetilde{m}$ and $\{0, ..., n-1\} = \widetilde{n}$ are approximately the same if $n/|Y| \approx m/|Y| \gg 0$. It means that if we plot the points $(n/|Y|, A_n)$ on the coordinate plane, we obtain the graph of a function continuous on $(0,\infty)$ (Theorem 6). The behavior of this function in the neighborhood of the origin is more complicated. In the paper the example of an S-integrable function, for which there exist very big intervals $\widetilde{m},\widetilde{n}$ such that $n/|Y|\approx m/|Y|\approx 0$, but $A_n\not\approx A_m$ is constructed. However, Theorem 7 shows stabilization of ergodic means on some initial segment of very big moments. In other words there exists a very big number m such that for all very big numbers $n<m$ one has $A_n\approx A_m$ for almost all $y\in Y$, i.e. the share of those $y\in Y$, for which the statement is not true, is infinitesimal.

Theorem 7 can be monitored in computer experiments in which the "very big" number $|Y|$ is not really too big ($\sim 50,000$). This theorem is proved with the help of classical ergodic theorem for Loeb spaces. So, we have an example how continuous mathematics helps to prove NSA theorem about hyperfinite objects, that has a substantive sense and can be monitored in computer experiments. On the other hand this theorem can hardly be formulated in the framework of the Cantor's Set Theory, were for the notion of a very big finite sets a sequence of finite sets, whose cardinalities tend to infinity is used. The transformation of Theorem 7 in a statement of classical mathematics is possible, since this theorem is a theorem of E. Nelson's Internal Set Theory [**?**]. However the obtained standard statement contains so many changes of quantifiers that is practically unreadable.