Lecture 2. Blackbody radiation and Planck’s formula.

September 17, 2006

A blackbody is an object that is perfect absorber of radiation. In ideal case it absorbs all light that falls on it – no light is reflected by it and no light passes through it. However, being heated it can radiate light. The first problem, where classical approach failed is the problem of explaining the thermal radiation of a blackbody.

A very rough model of a blackbody is the following. Each atom is tightly fixed by some forces at some equilibrium position. If it is slightly displaced from this position (e.g. by heating) it begin’s to oscillate like harmonic oscillator. Since all directions of displacement are possible each atom can be considered as three independent harmonic oscillators. Thus $N$ atoms form $N$ oscillators.

Recall that a harmonic oscillator is a system with the potential energy $U(x) = \frac{\gamma x^2}{2}$. Thus the full energy is given by

$$H(x, p) = \frac{p^2}{2m} + \frac{\gamma x^2}{2}.$$  \hspace{1cm} (1)

The displacement $x$ satisfies the differential equation $m\ddot{x} = -\gamma x$ (see equations (1) and (2) of Lecture 1), whose solution is given by $x = A\cos(\omega t + \varphi)$, where $\omega = \sqrt{\frac{\gamma}{m}}$. Thus the period $\tau = \frac{2\pi}{\omega}$ and the frequency $\nu = \frac{1}{\tau} = \frac{\omega}{2\pi}$. If the displacement is caused by heating with the absolute temperature $T$, then the density of the distribution of a certain oscillator on the phase plane $(x, p)$ is given by

$$f(x, p) = Ae^{-\frac{H(x, p)}{kT}},$$  \hspace{1cm} (2)

where $k = 8.31 \cdot 10^{-16}$ erg/degree is the Boltzman’s constant. These formula is obtained in classical thermodynamics. It means that the probability to find the oscillator in the rectangular $[x, x+dx] \times [p+dp]$ of the phase plane for infinitesimal $dx$ and $dp$ is equal to $f(x, p)dxdp$, where $f(x, p)$ satisfies (2).

Thus, the constant $A$ can be found from the condition

$$A\int\int e^{-\frac{H(x, p)}{kT}} dxdp = 1$$  \hspace{1cm} (3)

The average energy $\bar{H}$ (the mathematical expectation of energy) of a harmonic oscillator in a blackbody under the temperature $T$ can be found by

$$\bar{H} = \frac{\int\int H(x, p)e^{-\frac{H(x, p)}{kT}} dxdp}{\int\int e^{-\frac{H(x, p)}{kT}} dxdp}. $$  \hspace{1cm} (4)
This average energy can be easily found using the formula
\[ \int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi} \]

**Exercise 1.** Show that \( \overline{H} = kT \).

*Hint.* Set \( I(\alpha) = \iint e^{-\alpha H(x, p)} dxdp \). Then \( I'(\alpha) = -\iint H(x, p)e^{-\alpha H(x, p)} dxdp \). Thus, \( \overline{H} = -\frac{I'(\frac{1}{kT})}{I(\frac{1}{kT})} \).

Simply put, the spectrum of the radiation emitted by a blackbody under the temperature \( T \) is the distribution of energy by frequencies, which is measured by the density \( \rho(\nu, T) \). So, the energy of radiation of the unit of volume in the interval of frequencies \([\nu, \nu + d\nu]\) for an infinitesimal \( d\nu \) is equal to \( \rho(\nu, T)d\nu \).

It follows from the Maxwell’s theory of electricity, that

\[ \rho(\nu, T) = \frac{8\pi\nu^2}{c^3} \overline{H}, \tag{5} \]

where \( c = 3 \cdot 10^{10} \text{cm/sec} \) is the velocity of light.

Substituting \( kT \) for \( \overline{H} \) in (5) (see exercise 1), one obtains the Reyleigh-Jeans formula

\[ \rho(\nu, T) = \frac{8\pi\nu^2}{c^3} kT, \tag{6} \]

which immediately leads to a contradiction, since it follows from (6), that the energy of the radiation emitted by the unit of volume of a blackbody is infinite.

Actually, the Reyleigh-Jeans formula works only for small frequencies. W. Wien found experimentally the following formula for \( \rho(\nu, T) \) for big frequencies:

\[ \rho(\nu, T) = A\nu^3 e^{-\lambda\nu T}, \tag{7} \]

for certain parameters \( A \) and \( \lambda \).

Trying to find the general law (that works for all frequencies), M. Planck first of all found a formula, that is asymptotically equal to the Reyleigh-Jeans formula for small frequencies an to the Wien’s formula for big frequencies. This is the formula

\[ \rho(\nu, T) = \frac{A\nu^3}{e^{\frac{h\nu}{kT}}-1}, \]

**Exercise 2** Prove that this formula satisfies the required conditions.

Trying to explain this formula theoretically Planck concentrated on the term \( \overline{H} \) in the formula (5).
Using the formula (4) one can obtain the following simpler formula for $H$

$$H = \frac{\int_0^{\infty} E e^{-\frac{E}{kT}} dE}{\int_0^{\infty} e^{-\frac{E}{kT}} dE}$$

(8)

M. Planck suggested to replace integrals in this formula by discrete sums with the width $\varepsilon = h\nu$

**Remark 1.** This at first glance strange suggestion can be explained from the point of view (typical for many physicists) that actually we deal in real life with big sums of small amounts, as well as with the ratios of very small amounts and derivatives and integrals are only approximations convenient for calculations. From this point of view the idea that integral approximation may be irrelevant for the quantities in these problems looks natural.

In this case we obtain the following formula

$$H = \frac{\sum_{n=0}^{\infty} n h\nu e^{-\frac{nh\nu}{kT}}}{\sum_{n=0}^{\infty} e^{-\frac{nh\nu}{kT}}}$$

(9)

Using the same trick for series as we used for integrals in Exercise 1 and substituting the obtained expression for $H$ in the formula (5) it is easy to obtain the Planck’s formula for the blackbody radiation:

$$\rho(\nu, T) = \frac{8\pi h}{kT c^3} \cdot \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1}$$

(10)

**Exercise 3.** Perform the necessary calculations.

This formula perfectly agrees with experiment after determination (by an experiment) of the constant $h$. This is the famous Planck constant

$$h = 6.62 \cdot 10^{-34} \text{ joules/second}.$$ 

Sometimes it is convenient to use $\hbar = \frac{h}{2\pi}$.

So, the energy of an oscillator can assume only quantized values

$$E = nh\nu, \quad n = 0, 1, 2, 3, \ldots$$

(11)

Actually, this fact does not contradict classical mechanics. It follows from the law of conservation of energy that trajectories of an oscillator are the curve levels of the curves $H(p, q) = E$ for various constants $E$

**Exercise 4.** Deduce the law of the conservation of energy from the Hamilton equations (equations (4) of Lecture 1).

According to M. Planck the trajectories of oscillators are classical (ellipses in our case - see formula (1), but the initial conditions can be chosen only in such a way, that the energy on them assumes only one of the values given by (11). Actually, this suggestion is not true. Moreover, the modern quantum mechanics gives another formula for quantized values of energy

$$E = (n + \frac{1}{2})h\nu.$$ 

(12)

We will prove it later.