The goal of these lectures is to discuss mathematical foundations of quantum mechanics.

The mathematically rigorous exposition of other mathematical theories is usually based on some simple axioms, that clearly express certain physical laws obtained experimentally. Consider the following examples.

1. Classical mechanics

According to the second Newton’s law, the motion of a system of particles $M_1, \ldots, M_k$ with the coordinates $(x_1, y_1, z_1), \ldots, (x_k, y_k, z_k)$ under exertion of forces $F_i(x_1, y_1, z_1, \ldots, x_k, y_k, z_k, t)$ is determined by the following system of differential equations

$$m_i \ddot{x}_i = F_i,$$

where $x_i = (x_i, y_i, z_i)$, $m_i$ are masses of the particles, $i = 1, \ldots, k$, and $t$ is time.

Due to the existence and uniqueness theorem for differential equations the motion of this system is uniquely determined if the values $x(t_0)$, $v(t_0) = \dot{x}(t_0)$ are given for only one arbitrary moment of time $t = t_0$.

This is the reason to introduce the $6k$-dimensional space (or manifold) of points $(x, v)$ and describe the motion of the system by trajectories in this space. By some reasons it is more convenient to consider the momentum vector $p_i = m_i v_i$, where $v_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i)$ and $m_i$ is the mass of the point $M_i$, $i = 1, \ldots, k$.

The space of points $(x, p)$ is called the phase space of our system. Points of a phase space are called states. Usually the quantities that we measure in experiments such as coordinates, momenta, angular momenta, kinetic energy, total energy, etc. are some real-valued functions defined on the phase space. These functions are called observables.

For example, in the most interesting for us case of potential forces, i.e. when

$$\mathbf{F} = -\text{grad}V(x) = -\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial z_k}\right),$$

where $\mathbf{F}$ is $3k$-dimensional vector $<F_1, \ldots, F_k>$, the total energy $H$ is given by the formula

$$H(x, p) = \sum_{i=1}^{k} \frac{p_i^2}{2m_i} + V(x).$$

The first term in this sum $\sum_{i=1}^{k} \frac{p_i^2}{2m_i}$ is called the kinetic energy and is denoted by $W$, the second term $V(x)$ is called the potential energy or the potential of the system.
Put \( n = 3k \) and let us use letter \( q_i \) for each of the coordinates \( x_1, \ldots, x_k \) and letter \( p_i \) for the corresponding components of momentum vectors, i.e. \( p_i = m_i \dot{q}_i \). Then it is easy to see that the system (1) under condition (2) is equivalent to the following system

\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, \ldots, n.
\]  

(4)

Exercise. Prove it.

The system of equations (4) is called a Hamiltonian system.

Remark 1. It is not necessary that the coordinates \( q_i \) are 3-dimensional cartesian coordinates of particles. They may be obtained from cartesian coordinates by a change of variables. The motion of a system may take place not in 3-dimensional space, but in the plane or in any manifold. However, any system that can be described by \( 2n \) coordinates, satisfying equations (4) is called a Hamilton system. Thus, mathematically the classical mechanics (at least its significant part) is the theory of Hamilton systems.

2. Statistical mechanics (Thermodynamics).

Studying the thermodynamics of a gas or a rigid body we deal with systems that contain so enormously many particles (atoms or molecules) that it is impossible to fix all their coordinates at any given moment. In this case we can only use probability distributions \( \varphi(q,p) \) on the phase space \( \mathcal{M} \) of a given system. These distributions have to be obtained either experimentally or deduced from some experimental laws. For example, thermodynamics can be developed on the base of classical mechanics of many particles due to the Boltzman’s discovery that the temperature of a gas in the thermodynamic equilibrium is proportional to the average kinetic energy of molecules. Given a distribution \( \varphi(q,p) \) one can find the average value \( \bar{W} \) of any observable \( W(q,p) \) by the formula:

\[
\bar{W} = \int_{\mathcal{M}} W(q,p)dqdp
\]  

(5)

If we know a distribution \( \varphi(p,q) \) for a Hamilton system (4) at the initial moment \( t = t_0 \), then we can find the distribution \( \varphi_t(p,q) \) at any arbitrary moment \( t \) from the differential equation

\[
\frac{d\varphi_t}{dt} = -\{H, \varphi_t\},
\]  

(6)

where \( \{f,g\} \) is the Poisson bracket of functions \( f \) and \( g \) on \( \mathcal{M} \) defined by

\[
\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i}.
\]  

(7)

In the same way electrodynamics is based on the Maxwell’s equations, which can be considered as a mathematical formalization of Faraday laws obtained by him experimentally. Relativistic mechanics can be developed from an axiom that states that the velocity of light is the same in all moving coordinate systems. Though this statement looks controversially and even inconsistently
from the point of view of physics, it is very clear mathematically and the main result of relativistic theory follow from it very explicitly.

The situation with mathematical foundations of quantum mechanics looks very unlike those described above.

3. Quantum mechanics.

Mathematical formalization of quantum mechanics is based on the following principles. Similar to classical statistical mechanics, in quantum mechanics we deal with observables, whose averages one can measure in various states. However, the space of states is absolutely different from the previous cases and does not have an explicit physical sense.

The states in quantum mechanics are unit vectors in the separable complex Hilbert space $\mathcal{H}$. Moreover, we identify states $\psi$ and $e^{i\alpha}\psi$ for an arbitrary real $\alpha$. Thus, the space of states is the complex projective space over $\mathcal{H}$.

The observables are self-adjoint operators in the Hilbert space $\mathcal{H}$. The average $\overline{A}_\psi$ of an observable $A$ at a state $\psi$ is given by the formula

$$\overline{A}_\psi = (A\psi, \psi)$$ (8)

**Remark 2.** In literature on physics the notation $\langle \psi | A | \varphi \rangle$ is usually used for the dot product $(A\psi, \varphi)$. This notation comes up to P. Dirac. We will not use or discuss it here.

We see that, usual geometric space-time and other classical objects are not involved in the formulation of these basic principles. However, we speak about particles in terms of coordinates, momenta, energy, etc. Thus, we have to assign quantum observables to classical ones.

We will see that the uncertainty principle of quantum mechanics implies the unique (up to isomorphism) assignment of quantum observables to coordinates and momenta.

Consider, for example the case of $n = 1$. And represent the space $\mathcal{H}$ by $L^2(\mathbb{R})$. Then, the operators $Q$ and $P$ corresponding to the coordinate $q$ and linear momentum $p$ are given by

$$(Q\psi)(x) = x\psi(x), \quad (P\psi)(x) = \frac{h}{i} \frac{d\psi(x)}{dx},$$ (9)

where $\psi \in L^2(\mathbb{R})$ and $h$ is the Plank constant.

Even after formulas (9) are established, it is not obvious how to assign an operator $f(Q, P)$ to an arbitrary classical observable $f(q, p)$. It is well known how to define operator $g(A)$ for an arbitrary complex-valued real function of one variable $g(v)$ and for an arbitrary self-adjoint operator $A$ (and for the case of matrices – for an arbitrary matrix $A$). It is easy, further, to define $f(A, B)$, if the self-adjoint operators $A$ and $B$ commute, since in the case it can be proved that

$$A = g(C), \quad B = h(C)$$

for some real functions $h$ and $g$ and some self-adjoint operator $C$. The problem is that the operators $Q$ and $P$ do not commute. Indeed, they satisfy the famous Heisenberg relation:

$$[P, Q] = \frac{h}{i} I,$$ (10)

where $[A, B] = AB - BA$ is the commutator of $A$ and $B$, and $I$ is the identical operator.
Exercise. Prove it.

This makes it unclear, what operator should be assigned even to the function $p \cdot q$. Actually, this problem has not many solutions. Usually it is discussed under the following principle of correlation (between quantum and classical mechanics). The Plank constant $h$ is considered as a parameter. The operator $\frac{1}{h}[A, B]$ is denoted by $\{A, B\}_h$ and is called the quantum Poisson bracket. Let $S_h$ be a family of one-to-one linear mappings depending on the parameter $h$ from the space of functions on a classical phase space $\mathcal{M}$. For any two functions $f(p, q)$ and $g(p, q)$ on $\mathcal{M}$ denote by $f \ast_h g$ the function on $\mathcal{M}$ such that $S_h(f \ast_h g) = S_h(f) \circ S_h(g)$ and denote by $\{f, g\}_h$ the function on $\mathcal{M}$ such that $S_h(\{f, g\}_h) = \{S_h(f), S_h(g)\}_h$.

The family $S_h$ is called a quantization if for every functions $f$ and $g$ the following conditions hold:

$$f \ast_h g \xrightarrow{h \to 0} f \cdot g, \quad \{f, g\}_h \xrightarrow{h \to 0} \{f, g\}$$  \hspace{1cm} (11)

The conditions (11) express mathematically the following statement: ”The classical mechanics is the limit of quantum mechanics as the Plank constant tends to zero”.

In the case of $\mathcal{M} = \mathbb{R}^{2n}$ several quantizations are known. The most important is the Weyl quantization, that will be discussed in this course. For more complicated phase spaces some particular cases and some more general algebraic approaches exist. Their discussion is the main topic of these lectures.