Midterm 1

1) (Circular ring) Solve the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, $-L < x < L$, subject to $\frac{\partial u}{\partial x}(-L,t) = \frac{\partial u}{\partial x}(L,t)$, $u(-L,t) = u(L,t)$, for $t > 0$ and the following initial condition:

$$u(x,0) = |x|, -L < x < L.$$

The general solution has the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{(-n\pi/L)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} e^{(-n\pi/L)^2 kt}.$$

First observe that $u(x,0) = |x|$ is an even function. This implies that all the b_n's are zero and

$$a_n = \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}dx = 2L\frac{\cos(n\pi) - 1}{n^2\pi^2} = \begin{cases} \frac{-4L}{n^2\pi^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Also, $a_0 = \frac{1}{2L}\int_{-L}^{L} |x|dx = \frac{L}{2}$. So the solution is

$$u(x,t) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{-4L}{(2n-1)^2\pi^2} \cos\frac{(2n-1)\pi x}{L} e^{(-(2n-1)\pi/L)^2 kt}.$$

2) a) Check that $u(x,t) = 2kat + ax^2 + c_1x + c_0$ is a solution to $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$, where a, c_1, and c_0 are constants and k is the thermal diffusivity.

Check that $\frac{\partial u}{\partial t} = 2ka$, $\frac{\partial u}{\partial x} = 2ax + c_1$, and $k\frac{\partial^2 u}{\partial x^2} = 2ka$. So $u(x,t) = 2kat + ax^2 + c_1x + c_0$ is a solution to $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$.

b) Now consider the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ($k = 1$) for a thin, uniform rod with constant thermal properties and length 1 ($0 < x < 1$). Use part a) to provide a solution to the heat equation, in this situation, subject to the following boundary conditions and initial temperature distributions:

i) $\frac{\partial u}{\partial x}(0,t) = 0$, $\frac{\partial u}{\partial x}(1,t) = 1$, and $u(x,0) = \frac{1}{2}x^2 + 100$.

Since $\frac{\partial u}{\partial x} = 2ax + c_1$, $\frac{\partial u}{\partial x}(0,t) = 0$ implies that $c_1 = 0$ and $\frac{\partial u}{\partial x}(1,t) = 1$ implies that $a = \frac{1}{2}$. Finally, we use $u(x,0) = \frac{1}{2}x^2 + 100$ to see that $c_0 = 100$. So $u(x,t) = t + \frac{1}{2}x^2 + 100$ is the desired solution.

ii) $\frac{\partial u}{\partial x}(0,t) = 1$, $\frac{\partial u}{\partial x}(1,t) = 3$, $u(x,0) = x^2 + x$.

Since $\frac{\partial u}{\partial x} = 2ax + c_1$, $\frac{\partial u}{\partial x}(0,t) = 1$ implies that $c_1 = 1$ and then $\frac{\partial u}{\partial x}(1,t) = 3$ implies that $a = 1$. Finally, we use $u(x,0) = x^2 + x$ to see that $c_0 = 0$. So $u(x,t) = 2t + x^2 + x$ is the desired solution.