Homework Done Right

1.8) Prove that the intersection of any collection of subspaces of V is a subspace of V.

PROOF. Let Ω be an indexing set such that U_α is a subspace of V, for every $\alpha \in \Omega$, and let I be the intersection of these subspaces, that is, $I = \cap_{\alpha\in\Omega} U_\alpha$.

Since the U_α's are all subspaces, $0 \in U_\alpha$ for every $\alpha \in \Omega$, and so, $0 \in I$. Let x and y belong to I; this means that x and y are in every U_α. It follows that $x + y \in U_\alpha$, for every $\alpha \in \Omega$, because the subspaces are closed under addition. This demonstrates that I is closed under addition as $x+y \in U_\alpha$, for every $\alpha \in \Omega$, implies that $x+y \in I$. Now let $a \in F$ and let $x \in I$. The U_α's, being subspaces, are closed under scalar multiplication, and so $ax \in U_\alpha$, for every $\alpha \in \Omega$; therefore, $ax \in I$ and I is closed under scalar multiplication.

□

1.14) Suppose that U is a subspace of $P(F)$ consisting of all polynomials p of the form $p(z) = az^2 + bz^5$, where $a, b \in F$. Find a subspace W of $P(F)$ such that $P(F) = U \oplus W$.

Observe that, by definition, $U = span(z^2, z^5)$, and so, U is a subspace of $P(F)$. Also, note that $(1, z, z^2, z^3, z^4, ...)$ is a basis for $P(F)$. Now let $W = span(1, z, z^3, z^4, z^6, z^7, z^8, ...)$; this means W is also a subspace of $P(F)$.

If $U \cap W$ contains some nonzero polynomial such as $az^2 + bz^5$, where $a, b \in F$, then we get that $az^2 + bz^5 \in span(1, z, z^3, z^4, z^6, z^7, z^8, ...)$. This is impossible as $(1, z, z^2, z^3, z^4, ...)$ are linearly independent and so no vector in the list may be expressed as a linear combination of other vectors in the list. It follows that $U \cap W = \{0\}$.

Now $U + W$ contains all the vectors in the list $(1, z, z^2, z^3, z^4, ...)$, which is a basis for $P(F)$, and thus, $U + W$ contains all linear combinations of the basis vectors. This implies that $U + W$ contains all vectors in $P(F)$; so $P(F) \subset U + W$. Since U and W are subspaces of $P(F)$, $U + W$ is a subspace of $P(F)$; so $U + W \subset P(F)$. The two inclusions of the sets obtained prove that $U + W = P(F)$. This combined with $U \cap W = \{0\}$ shows that $P(F) = U \oplus W$.

2.1) Prove that if $(v_1, v_2, ..., v_n)$ spans V, then so does the list $(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$.

PROOF. Let $w \in V$. So $w = a_1v_1 + a_2v_2 + ... + a_nv_n$, for some $a_1, a_2, ..., a_n \in F$. We can rewrite the expression as $w = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + ... + (a_1 + a_2 + ... + a_n)v_n$. This shows that $w \in span(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$.

Conversely, suppose that $v \in span(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$, that is, $v = a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + ... + a_nv_n$, for some $a_1, a_2, ..., a_n \in F$. Then $v = a_1v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + ...(a_{n-1} - a_{n-2})v_{n-1} + a_nv_n$, which is certainly in $span(v_1, v_2, ..., v_n)$. This shows that the list $(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n)$ spans V.

□

2.10) Suppose V is finite dimensional, with $\dim V = n$. Prove that there exist one-dimensional subspaces $U_1, U_2, ..., U_n$ of V such that $V = U_1 \oplus U_2 \oplus ... \oplus U_n$.

PROOF. Let $(v_1, v_2, ..., v_n)$ be a basis for V, and let $U_i = span(v_i)$, for $i \in \{1, 2, ..., n\}$. Since the v_i's are not zero, by definition, the U_i's are one-dimensional subspaces of V.

$U_1 + U_2 + ... + U_n$ contains every vector in the basis, and thus, it contains every vector that can be expressed as a linear combination of the basis elements. So $V \subset U_1 + U_2 + ... + U_n$, and thus, $V = U_1 + U_2 + ... + U_n$ since $U_1 + U_2 + ... + U_n$ is a subspace of V.

Suppose $v \in U_i \cap U_j$, for $i, j \in \{1, 2, ..., n\}$ and $i \neq j$. This implies that $v = av_i = bv_j$, where $a, b \in F$. Since v_i, v_j are linearly independent, we get that $a = b = 0$, and so $v = 0$. It follows that $U_i \cap U_j = \{0\}$, for $i, j \in \{1, 2, ..., n\}$ and $i \neq j$. This, together with the fact that $V = U_1 + U_2 + ... + U_n$, proves that $V = U_1 \oplus U_2 \oplus ... \oplus U_n$.

□

3.16) Suppose that U and V are finite-dimensional and $S \in L(V, W)$, $T \in L(U, V)$. Prove that $\dim null(ST) \leq \dim null(T) + \dim null(S)$.

PROOF. First, some notation. Let T' be the linear map T restricted to the subspace $null(ST)$, that is, $T'u = Tu$ whenever $u \in null(ST)$ and T' is not defined otherwise.

Observe that if $u \in null(T')$, then $T'u = Tu = 0$. So $u \in null(T)$ and we get the simple inclusion $null(T') \subset null(T)$. This implies that $\dim null(T') \leq \dim null(T)$.

Also, if $u \in null(ST)$, then $(ST)u = 0 = S(Tu) = S(T'u)$. This implies that $T'u \in null(S)$, and hence, $range(T') \subset null(S)$. From this we obtain our second inequality $\dim range(T') \leq \dim null(S)$.

Now, the dimension formula, Theorem 3.4, applied to $null(ST)$ and T' says $\dim null(ST) = \dim null(T') + \dim range(T')$. Combining this with the inequalities $\dim null(T') \leq \dim null(T)$ and $\dim range(T') \leq \dim null(S)$, we get the required result $\dim null(ST) \leq \dim null(T) + \dim null(S)$.

□

5.11) Suppose $S, T \in L(V)$. Prove that ST and TS have the same eigenvalues.

PROOF. Let λ be an eigenvalue for ST, that is $ST(v) = \lambda v$, for some nonzero $v \in V$. Apply T to both sides. $T(ST(v)) = (TS)(Tv) = \lambda Tv$. Now, if $Tv \neq 0$, then Tv is an eigenvector with eigenvalue λ for TS. If $Tv = 0$, then $\lambda = 0$ and T is not invertible, and hence, TS is not invertible, which implies that TS has zero as an eigenvalue since its null space is not trivial. So in all cases, if λ is an eigenvalue for ST, it is also an eigenvalue for TS. A completely symmetric argument shows that if λ is an eigenvalue for TS, it is also an eigenvalue for ST. It follows that ST and TS have the same eigenvalues.

□

5.14) Suppose V is finite-dimensional and $T \in L(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in L(V)$.

PROOF. If $T = \lambda I$ for some $\lambda \in F$, then $ST = S\lambda I = \lambda SI = \lambda IS = TS$, for any $S \in L(V)$.

Suppose $ST = TS$ for every $S \in L(V)$. Let $(v_1, ..., v_n)$ be a basis. $Tv_1 = a_1 v_1 + ... + a_n v_n$, for some $a_1, ..., a_n \in F$, because the image of v_1 is some vector in V, and hence, a linear combination of the basis elements.

Now define a linear map S by describing what it does to the basis elements. $S(v_1) = v_1$ and S sends all other basis vectors to zero. Then look at $ST(v_1) = TS(v_1)$. The left side of the equation is $a_1 v_1$ and the right side is Tv_1. So $Tv_1 = a_1 v_1$. The same argument shows that $Tv_k = a_k v_k$, for $k \in \{1, ..., n\}$.

We must show that all these a_k's are the same. Again, define a linear map S by describing what it does to the basis elements. $S(v_1) = v_2, S(v_2) = v_1$ and S sends all other basis vectors to zero. Then look at $ST(v_1 + v_2) = TS(v_1 + v_2)$. The left side of the equation is $a_2 v_1 + a_1 v_2$ while the right side is $a_1 v_1 + a_2 v_2$. Since these are equal and because v_1, v_2 are linearly independent, $a_1 = a_2$. Similar arguments show that $a_i = a_j$, for $i, j \in \{1, ..., n\}$.

Since all these a_k's are the same, we may rename them and call them λ. So we have shown that $Tv_i = \lambda v_i$, for $i \in \{1, ..., n\}$. Let v be a vector in V, that is $v = a_1 v_1 + ... + a_n v_n$, for some $a_1, ..., a_n \in F$. Now $Tv = T(a_1 v_1 + ... + a_n v_n) = \lambda a_1 v_1 + \lambda a_2 v_2 ... + \lambda a_n v_n = \lambda(a_1 v_1 + ... + a_n v_n) = \lambda v$. This shows that T is a scalar multiple of the identity.

$\square$

5.21) Suppose $P \in L(V)$ and $P^2 = P$. Prove that $V = null(P) \oplus range(P)$.

PROOF. V is a direct sum of $null(P)$ and $range(P)$ if the intersection of $null(P)$ and $range(P)$ is just zero and every vector in V can be written as a sum of a vector in $null(P)$ with a vector in $range(P)$.

Suppose x is in both $null(P)$ and $range(P)$. Because x is in $null(P)$, $P(x) = 0$. But x is also in $range(P)$ so there is a y in V such that $Py = x$. Since $P^2 = P$, $P^2 y = Py$, and also $P^2(y) = P(Py) = Px$. It follows that $x = Py = P^2 y = Px = 0$. This gives us that $null(P) \cap range(P) = \{0\}$.

Let v be a vector in V. Then $v = Pv + (v - Pv)$. Now, $Pv \in range(P)$, by definition, and $P(v - Pv) = Pv - P^2 v = 0$, because $P^2 = P$, implies that $(v - Pv) \in null(P)$. So every vector in V can be written as a sum of a vector in $null(P)$ with a vector in $range(P)$.

Since we have checked these two conditions, we have proved that $V = null(P) \oplus Range(P)$. $\square$

6.17) Prove that if $P \in L(V)$, $P^2 = P$, and every vector in $null(P)$ is orthogonal to every vector in $range(P)$, then P is an orthogonal projection.

PROOF. By the previous exercise, $V = null(P) \oplus range(P)$ and since every vector in $null(P)$ is orthogonal to every vector in $range(P)$, $null(P) = (range(P))^{\perp}$. Now let $v \in V$, and so $v = u + n$, with $u \in range(P)$ and $n \in null(P)$. Also, $v = Pv + (v - Pv)$, with $Pv \in range(P)$ and $(v - Pv) \in null(P)$. Because $V = null(P) \oplus range(P)$, the representation of v as a sum of a vector in $range(P)$ with a vector in $null(P)$ is unique, and therefore, $Pv = u$.

So, by definition, P is a projection of V onto $range(P)$ and an orthogonal projection since $null(P) = (range(P))^{\perp}$. □

6.2) Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0$ if and only if $||u|| \leq ||u + av||$ for all $a \in F$.

PROOF. $||u|| \leq ||u + av|| \Leftrightarrow ||u||^2 \leq ||u + av||^2 \Leftrightarrow \langle u, u\rangle \leq \langle u + av, u + av\rangle$

$\Leftrightarrow \langle u, u\rangle \leq \langle u, u\rangle + \overline{a}\langle u, v\rangle + a\overline{\langle u, v\rangle} + a\overline{a}\langle v, v\rangle \Leftrightarrow$

$-2Re\{a\langle u, v\rangle\} \leq a\overline{a}\langle v, v\rangle$. We will only work with the last inequality.

If $\langle u, v\rangle = 0$, the inequality holds since the L.H.S is zero and R.H.S is always greater than or equal to zero.

Now suppose that $||u|| \leq ||u + av||$ is true for all a in F. So we know that $-2Re\{a\langle u, v\rangle\} \leq a\overline{a}\langle v, v\rangle$ also holds for all a in F. Suppose $\langle u, v\rangle \neq 0$. Write $a = -b\overline{\langle u, v\rangle}$, where b is a positive real. Note that $-2Re\{a\langle u, v\rangle\} = 2b|\langle u, v\rangle|^2$ and $a\overline{a}\langle v, v\rangle = b^2|\langle u, v\rangle|^2\langle v, v\rangle$. So the inequality reduces to $2b|\langle u, v\rangle|^2 \leq b^2|\langle u, v\rangle|^2\langle v, v\rangle$. Since b is positive and $|\langle u, v\rangle|^2 \neq 0$, we can divide by $b|\langle u, v\rangle|^2$, preserving the inequality to get $2 \leq b\langle v, v\rangle$. Now we can make the R.H.S as small as we want by letting b go to zero. When the R.H.S is smaller than 2, we get a contradiction. So $\langle u, v\rangle = 0$.

□

6.18) Prove that if $P \in L(V)$, $P^2 = P$, and $\|Pw\| \leq \|w\|$ for all $w \in V$, then P is an orthogonal projection.

PROOF. By 6.17, it is sufficient to show that every vector in $null(P)$ is orthogonal to every vector in $range(P)$. First, observe that if $u \in range(P)$, then $Pu = u$. This is because there is some $x \in V$ such that $Px = u$, and then $Pu = P^2x = Px = u$.

Now, let $v \in null(P)$ and consider the vector $w = u + av$, with $a \in F$. Note that $Pw = Pu + aPv = u$ and so $\|Pw\| = \|u\|$. The given inequality $\|Pw\| \leq \|w\|$ gives us $||u|| \leq ||u + av||$, where a was some arbitrary scalar. It follows that $||u|| \leq ||u + av||$ for all $a \in F$. By exercise 6.2, we get that u and v are orthogonal. Here u was some arbitrary vector in $range(P)$ and v was some arbitrary vector in $null(P)$. So we may conclude that every vector in $null(P)$ is orthogonal to every vector in $range(P)$, which is sufficient to show that P is an orthogonal projection.

$\square$