Equivalent trace sets for arithmetic Fuchsian groups

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Abstract

We show that the modular group has an infinite family of finite index subgroups, each of which has the same trace set as the modular group itself. Various congruence subgroups of the modular group, and the Bianchi groups, are also shown to have this property.

1 Introduction

For a Riemannian manifold M, the *eigenvalue spectrum* is the set of eigenvalues of the Laplacian operator, and the *length spectrum* is the set of lengths of closed geodesics, both counted with multiplicity. The two spectra are closely related, and together determine much about the manifold: though constructions such as Sunada's [14] show that there exist isospectral, non-isometric manifolds, it is known that manifolds for which these spectra are equal must share certain geometric and topological properties; for example, if the manifolds are hyperbolic, they must have the same volume [12]. It is also possible to define the *eigenvalue set* $E(M)$ and the *length set* $L(M)$ to be the respective spectra with multiplicities discarded. It is known that these form invariants which are considerably coarser; for example, Leininger, McReynolds, Neumann, and Reid [9] proved that if M is a compact hyperbolic manifold, then there exist sequences of pairs of covers $\{M_i, N_i\}$ such that for all i, $E(M_i) = E(N_i)$ and $L(M_i) = L(N_i)$, but the ratio $\frac{\mathrm{Vol}(M_i)}{\mathrm{Vol}(N_i)}$ diverges to ∞.

When the manifold in question is a hyperbolic surface, so that the fundamental group $\pi_1(M)$ is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$, it is well-known that the length ℓ of a closed geodesic determines, and is determined by, (the absolute value of) the trace tr of the corresponding hyperbolic isometry via the equation

$$|\mathrm{tr}\,| = 2\cosh\frac{\ell}{2}.$$

There is a similar correspondence when M is a hyperbolic 3–manifold, so that $\pi_1(M) < \mathrm{PSL}_2(\mathbb{C})$ acts on the hyperbolic 3–space $\mathbb{H}^3$. Here a complex trace corresponds to a *complex length* in M; the corresponding action on $\mathbb{H}^3$ involves a combination of translation along and rotation around the axis of the isometry. In these cases, the length set (resp. complex length set) of the manifold $M = \mathbb{H}^n/\Gamma$ is in direct correspondence with the trace set of the Fuchsian (resp. Kleinian) group Γ.

It is therefore a consequence of the aforementioned result of Leininger, McReynolds, Neumann and Reid that there exist pairs of Fuchsian and Kleinian groups of different covolumes but with equal trace sets. One may then ask the question of, given a prescribed trace set, how many (if any) groups possess precisely that trace set. In particular, are there any Fuchsian or Kleinian groups which are uniquely determined by their trace set? In this direction, Schmutz [13] showed that there are infinitely many Fuchsian groups with the same trace set as certain congruence subgroups of the modular group $\mathrm{PSL}_2(\mathbb{Z})$. In a similar direction, the goal of this note is to determine to what extent $\mathrm{PSL}_2(\mathbb{Z})$, and certain subgroups thereof, are determined by their trace sets. It is a consequence of Takeuchi's characterization of arithmetic Fuchsian groups [15] that any (cofinite) Fuchsian group with trace set precisely the rational integers $\mathbb{Z}$ must be arithmetic, and in fact (conjugate to) a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ (see also Geninska and Leuzinger [7]). We show that there are infinitely many such finite index subgroups by constructing a finitely generated, infinite index subgroup H with the same trace set, and appealing to the fact that the modular group has the property of being *subgroup separable*, also called locally extended residually finite (LERF), which implies the existence of one (and hence an infinite descending chain of) finite index subgroup(s) containing H.

More generally, we show the following.

Theorem 1.1. *Let $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ be a cofinite Fuchsian group with trace set $\mathrm{tr}\,(\Gamma)$. Let $G_1, \ldots, G_m$ be a finite collection of finitely generated, infinite index subgroups of Γ such that*

$$\bigcup_{i=1}^{m} \mathrm{tr}\,(G_i) = \mathrm{tr}\,(\Gamma),$$

and for each $1 \leq i \leq m$, let Q_i be a finite-sided, connected fundamental domain for G_i. Then there exist $\alpha_i \in \Gamma$ such that the subgroup H of Γ generated by the conjugates $\alpha_i G_i \alpha_i^{-1}$ is a subgroup of Γ of infinite index, with $\mathrm{tr}\,(H) = \mathrm{tr}\,(\Gamma)$.

Theorem 1.1 is then applied to $\mathrm{PSL}_2(\mathbb{Z})$, and to families of congruence subgroups thereof.

Corollary 1.2. *Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, or a congruence subgroup $\Gamma_0(n)$ or $\Gamma(n)$ for some $n \in \mathbb{N}$. Then there exists a finitely generated, infinite index subgroup $H_\Gamma < \Gamma$ with the same trace set as Γ. Hence, there exist infinitely many finite index subgroups of Γ with this trace set.*

The most natural lattices in $\mathrm{PSL}_2(\mathbb{C})$ which serve as analogues of the modular group are the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$, where $d > 0$ is a square-free integer, and $\mathcal{O}_d$ is the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$. We show a similar result for these groups.

Theorem 1.3. *Given any Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$, there are infinitely many finite index subgroups $\Gamma < \mathrm{PSL}_2(\mathcal{O}_d)$ with the same complex trace set as $\mathrm{PSL}_2(\mathcal{O}_d)$.*

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2 Preliminaries

We refer to Beardon [3] for more details of the contents of this section. We consider the upper half-plane and upper half-space models for hyperbolic 2- and 3-space $\mathbb{H}^2$ and $\mathbb{H}^3$ respectively. The group of conformal, orientation-preserving isometries (or linear fractional transformations) of $\mathbb{H}^2$ (resp. $\mathbb{H}^3$) can be identified with $\mathrm{PSL}_2(\mathbb{R})$ (resp. $\mathrm{PSL}_2(\mathbb{C})$) via the correspondence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \longleftrightarrow \quad z \longmapsto \frac{az+b}{cz+d}.$$

Given an element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$, the trace $\mathrm{tr}\,\gamma$ is not well-defined, but is well-defined up to sign. Given a discrete group $\Gamma < \mathrm{PSL}_2(\mathbb{C})$, we define the *trace set* of Γ to be

$$\mathrm{tr}\,(\Gamma) = \{\mathrm{tr}\,\gamma \mid \gamma \in \Gamma \setminus \{1\}\} / (x \sim -x).$$

For every $n \in \mathbb{Z}$, the modular group $\mathrm{PSL}_2(\mathbb{Z})$ has an element with trace n; taking the above into account, throughout this note we will say that $\mathrm{PSL}_2(\mathbb{Z})$ has trace set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If $|\mathrm{tr}\,\gamma| < 2$, then γ is elliptic and fixes a point of $\mathbb{H}^2$, or fixes an axis of $\mathbb{H}^3$ pointwise. If $|\mathrm{tr}\,\gamma| = 2$, then γ is parabolic and fixes exactly one point on the boundary circle or sphere. If $|\mathrm{tr}\,\gamma| > 2$, then γ is hyperbolic, fixes two points on the boundary circle or sphere, and acts as a translation along the geodesic between these two fixed points. If $\operatorname{tr}\gamma \notin \mathbb{R}$, then γ is loxodromic, fixes two points on the boundary, and acts as both a translation along, and a rotation around, the axis between the two fixed points.

We can study the action of an element $\gamma \in \mathrm{PSL}_2(\mathbb{C})$ which does not fix ∞ on the upper half-plane model for $\mathbb{H}^2$ or $\mathbb{H}^3$ by considering *isometric circles* or *isometric spheres*. Given

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{C}),$$

where $c \neq 0$ (since we assume γ does not fix ∞) the isometric sphere S_γ of γ has center $\frac{-d}{c}$ and radius $\frac{1}{|c|}$; furthermore, the isometric sphere $S_{\gamma^{-1}}$ has center $\frac{a}{c}$ and the same radius $\frac{1}{|c|}$. The action of γ on S_γ is by a Euclidean isometry, and γ sends the exterior E_γ (resp. interior I_γ) of S_γ to the interior $I_{\gamma^{-1}}$ (resp. exterior $E_{\gamma^{-1}}$) of $S_{\gamma^{-1}}$. In particular, we note that when $\operatorname{tr}\gamma = 0$, corresponding to a rotation of order 2, the isometric spheres S_γ and $S_{\gamma^{-1}}$ coincide, and γ then acts by exchanging the interior and exterior of S_γ. In the following, when we refer to a closure $\overline{P}$ of a set $P \subset \mathbb{H}^2$, we mean the closure taken in $\mathbb{H}^2 \cup \mathbb{R} \cup \infty$.

When Γ contains a parabolic element fixing ∞, the set of isometric circles of elements of Γ is invariant under this subgroup. In this case, one may construct a fundamental domain, called a *Ford domain*, by taking the set E of points exterior to all isometric circles, and intersecting it with a fundamental region for the stabilizer of ∞. When $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ is generated by (at most two) parabolics fixing ∞ and a single element γ which does not fix ∞, then we have two cases of how a Ford domain may be constructed. If the isometric spheres of γ do not intersect, or if $\operatorname{tr}\gamma \in \mathbb{R}$, then they (and their translates) suffice to form the boundary of a Ford domain. If not, then the isometric spheres of powers of γ may not be covered by those of γ, and so appear in the boundary of a Ford domain. In this case, we will use the properties that any isometric circle of $\gamma^{\pm n}$ contains one of the fixed points of γ, and that if γ belongs to a Bianchi group, then the isometric spheres have radius bounded above by 1.

Given two non-cofinite Fuchsian or Kleinian groups $G_1, G_2 < G$, the Klein–Maskit combination theorem gives a way of ensuring that the group $\langle G_1, G_2\rangle$ generated by these two subgroups inside of G is also non-cofinite. Precisely, it states (see Maskit [11], p. 139):

Theorem 2.1 (Klein–Maskit Combination Theorem)**.** *Suppose $G_1, G_2 < G$ have fundamental domains D_1 and D_2 respectively, and that $D_1 \cup D_2 = \mathbb{H}^n$ $(n = 2, 3)$ and $D_1 \cap D_2 \neq \emptyset$. Then $\langle G_1, G_2\rangle = G_1 * G_2$, and $D = D_1 \cap D_2$ is a fundamental domain for $G_1 * G_2$.*

A group G is called *residually finite* if for any non-trivial element $g \in G$, there is a finite index subgroup $K < G$ such that $g \notin K$. The group G has the stronger property of being *subgroup separable* (or LERF) if for any finitely generated subgroup $H < G$ and any $g \in G \setminus H$, there exists a finite index subgroup $K < G$ such that $H \subset K$ and $g \notin K$. Equivalently, G is LERF if every finitely generated subgroup $H < G$ is the intersection of finite index subgroups of G; we will appeal to this alternative formulation. The fact that $\mathrm{PSL}_2(\mathbb{Z})$ is LERF follows from the fact that it contains a free group of finite index, and Hall's result [8] that free groups are LERF; the fact that the Bianchi groups are LERF follows from work of Agol, Long and Reid [2], Agol [1], Calegari and Gabai [4], and Canary [5].

Given a natural number n, the *principal congruence subgroup* $\Gamma(n) < \mathrm{PSL}_2(\mathbb{Z})$ consists of those matrices which are congruent to the identity modulo n; that is

$$\Gamma(n) = \left\{ \begin{pmatrix} 1+an & bn \\ cn & 1+dn \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) \right\}.$$

All principal congruence subgroups are finite index and normal in $\mathrm{PSL}_2(\mathbb{Z})$, since they are the kernels of the natural surjective homomorphisms $\psi_n : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ given by reducing entries modulo n. A similar family of groups is given by the upper triangular congruence subgroups $\Gamma_0(n) < \mathrm{PSL}_2(\mathbb{Z})$; these consist of matrices which are congruent to upper triangular matrices modulo n. Any maximal arithmetic

Fuchsian group which is commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ is obtained by taking the normalizer $N(\Gamma_0(n))$ of $\Gamma_0(n)$ for n square-free, where the normalizer is taken in $\mathrm{PSL}_2(\mathbb{R})$.

3 Fuchsian groups

In this section, we establish that certain families of Fuchsian groups have the property that each contains infinitely many finite index subgroups with the same trace set as itself. This will be a consequence of the following more general result.

Theorem 3.1. *Let $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ be a cofinite Fuchsian group with trace set $\mathrm{tr}\,(\Gamma)$. Let $G_1, \ldots, G_m$ be a finite collection of finitely generated, infinite index subgroups of Γ such that*

$$\bigcup_{i=1}^{m} \mathrm{tr}\,(G_i) = \mathrm{tr}\,(\Gamma),$$

and for each $1 \leq i \leq m$, let Q_i be a finite-sided, connected fundamental domain for G_i. Then for each i there exists $\alpha_i \in \Gamma$ having isometric circle S_{α_i} with interior I_{α_i} such that:

- *for each $1 \leq i \leq m$, $\alpha_i \notin G_i$;*
- *for each $1 \leq i \leq m$, we have $\overline{I_{\alpha_i}} \subset \overline{Q_i} \subset \overline{\mathbb{H}^2}$; and*
- *for any $j \neq k$, we have $\overline{I_{\alpha_j^{-1}}} \cap \overline{I_{\alpha_k^{-1}}} = \emptyset$.*

Hence, the subgroup H of Γ generated by the conjugates $\alpha_i G_i \alpha_i^{-1}$ is a subgroup of Γ of infinite index, with $\mathrm{tr}\,(H) = \mathrm{tr}\,(\Gamma)$.

Proof. The existence of such α_i follows from the assumptions on the G_i as follows. Choose disjoint open intervals $(x_i, y_i) \subset \overline{Q_i} \cap \mathbb{R}$ and isometries $\alpha_i \in \Gamma$ such that for each i, $\overline{I_{\alpha_i}} \cap \mathbb{R}, \overline{I_{\alpha_i^{-1}}} \cap \mathbb{R} \subset (x_i, y_i)$. These α_i can for example be constructed by taking a hyperbolic isometry γ_i whose axis endpoints are in the relevant interval (which must exist by the assumption that Γ is cofinite), and taking a sufficiently high power for the α_i in order that the isometric spheres satisfy the required condition.

The subgroups $H_i := \alpha_i G_i \alpha_i^{-1}$, and their fundamental domains $\alpha_i(Q_i)$ have the properties that for any $j \neq k$, $\alpha_j(Q_j) \cup \alpha_k(Q_k) = \mathbb{H}^2$, and $\alpha_j(Q_j) \cap \alpha_k(Q_k) \neq \emptyset$. As such, the repeated application of Theorem 2.1, to H_1 and H_2, and then to $H_1 * H_2$ and H_3 etc., gives that the subgroup H generated by the H_i has a fundamental domain

$$Q = \bigcap_{i=1}^{m} \alpha_i(Q_i).$$

Since for each i, the complement $\mathbb{H}^2 \setminus \alpha_i(Q_i)$ is contained in $I_{\alpha_i^{-1}}$, and these $I_{\alpha_i^{-1}}$ are mutually disjoint, it follows that Q contains the intersection of the exteriors $E_{\alpha_i^{-1}}$, and thus has infinite area. This implies that $H < \Gamma$ is an infinite index subgroup. Finally, each trace of $\mathrm{tr}\,(\Gamma)$ also belongs to $\mathrm{tr}\,(H)$, and so $\mathrm{tr}\,(H) = \mathrm{tr}\,(\Gamma)$ as required. $\square$

The following Lemma will be helpful in applying Theorem 3.1 to specific examples.

Lemma 3.2. *Let G be a Fuchsian group generated by two elements of the form*

$$G = \left\langle g_1 = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle,$$

where $c \neq 0$, and suppose that $\frac{|a+d|}{c} < \frac{|m|}{2}$, and that $E_{g_2} \cap E_{g_2^{-1}}$ is a Ford domain for $\langle g_2 \rangle$. Then if $|m| > \frac{4}{|c|}$, the group G admits a finite-sided Ford fundamental domain of infinite area.

Proof. The hypothesis that $\frac{|a+d|}{c} < \frac{|m|}{2}$ implies that the centers of the isometric circles of g_2 and g_2^{-1} are at most $\frac{|m|}{2}$ apart, and since $|m| > \frac{4}{|c|}$, their most distant endpoints are at most $\frac{|m|}{2} + \frac{2}{|c|} < \frac{|m|}{2} + \frac{|m|}{2} = |m|$ apart. We set

$$F = \left\{ z \in \mathbb{H}^2 \mid \left| \mathrm{Re}(z) - \frac{a-d}{2c} \right| \leq \frac{|m|}{2} \right\},$$

which is a fundamental region for $\langle g_1 \rangle$, and note that the hypotheses imply that

$$Q = E_{g_2} \cap E_{g_2^{-1}} \cap F$$

is a fundamental domain for G, and has infinite area. Hence, we are done. $\square$

We now apply Theorem 3.1 to show that $\mathrm{PSL}_2(\mathbb{Z})$ has infinitely many finite index subgroups with the same trace set as itself.

Corollary 3.3. *The modular group* $\mathrm{PSL}_2(\mathbb{Z})$ *has a finitely generated, infinite index subgroup* H *with trace set* $\mathrm{tr}\,(H) = \mathbb{N}_0$. *Hence, there exist infinitely many finite index subgroups of* $\mathrm{PSL}_2(\mathbb{Z})$ *with this trace set.*

Proof. We apply Theorem 3.1 to the following subgroups:

$$G_0 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \right\rangle,$$

$$G_1 = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \right\rangle,$$

and

$$G_2 = \left\langle \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

Each subgroup satisfies the hypotheses of Lemma 3.2, and we may take for each fundamental domain Q_i the Ford domain bounded by the isometric spheres of the first generators and the vertical geodesics from -1 and 4 to ∞ respectively. Note that for each $0 \leq i \leq 2$, the closure $\overline{Q_i} \subset \mathbb{H}^2 \cup S_\infty$ contains the open interval $(3,4)$. For the conjugating elements, we take

$$\alpha_0 = \begin{pmatrix} 142 & -545 \\ 37 & -142 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 17 & -58 \\ 5 & -17 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 117 & -370 \\ 37 & -117 \end{pmatrix}.$$

Thus the subgroup H is generated by the elements

$$\begin{pmatrix} 26269 & -100820 \\ 6845 & -26271 \end{pmatrix}, \begin{pmatrix} -82644 & 317189 \\ -21533 & 82644 \end{pmatrix}, \begin{pmatrix} 424 & -1445 \\ 125 & -426 \end{pmatrix},$$

$$\begin{pmatrix} -782 & 2667 \\ -229 & 781 \end{pmatrix}, \begin{pmatrix} 21644 & -68445 \\ 6845 & -21646 \end{pmatrix}, \text{ and } \begin{pmatrix} -20241 & 64009 \\ -6400 & 20239 \end{pmatrix}.$$

We may now invoke the equivalent definition of LERF to see that this finitely generated, infinite index subgroup H must be the intersection of finite index subgroups of $\mathrm{PSL}_2(\mathbb{Z})$. There must be infinitely many of these finite index subgroups, and the trace set of each contains $\mathrm{tr}\,(H) = \mathbb{N}_0$, so we are done. $\square$

The existence of the subgroup H (and hence a descending chain of finite index subgroups with the same trace set) raises a number of questions. For example, this method does not give explicitly a finite index subgroup, and so we would like to know more about the structure of such subgroups.

Question. What is the minimal index for a finite index subgroup $H < \mathrm{PSL}_2(\mathbb{Z})$ with $\mathrm{tr}\,(H) = \mathbb{N}_0$?

It is possible to prove that the unique index 2 subgroup, and the two (conjugacy classes of) subgroups of index 3, do not have trace set $\mathbb{N}_0$, and so we can say that the minimal index must be at least 4. Also, this method also fails to produce a subgroup whose trace set is $\mathbb{N}_0 \setminus \{0, 1\}$, which would correspond to a manifold with the same trace set as $\mathrm{PSL}_2(\mathbb{Z})$ except for torsion elements.

Question. Does there exist a finite index subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ with trace set $\mathbb{N}_0 \setminus \{0, 1\} = \{2, 3, 4, \dots\}$?

It is possible to show the existence of infinitely many families of groups with the same trace set. The next two results show that congruence subgroups of certain forms have infinitely many finite index subgroups of the same trace set. In particular, this shows that there are examples of torsion-free groups with this property.

Corollary 3.4. *For each $n \in \mathbb{N}$, the congruence subgroup $\Gamma_0(n) < \mathrm{PSL}_2(\mathbb{Z})$ admits a finitely generated, infinite index subgroup $H_n < \Gamma_0(n)$ such that $\mathrm{tr}\,(H_n) = \mathrm{tr}\,(\Gamma_0(n))$.*

Proof. Consider the maps

$$\mathrm{PSL}_2(\mathbb{Z}) \xrightarrow{\pi} \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z}) \xrightarrow{\mathrm{tr}} \mathbb{Z}/n\mathbb{Z},$$

where $\pi : \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ denotes the natural projection where each entry is reduced modulo n, and tr is the trace map. For an element of $\Gamma_0(n)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xmapsto{\pi} \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{a}^{-1} \end{pmatrix} \xmapsto{\mathrm{tr}} \bar{a} + \bar{a}^{-1},$$

and thus the set $\{\bar{a} + \bar{a}^{-1} \mid \bar{a} \in \mathbb{Z}/n\mathbb{Z}$ has a multiplicative inverse$\}$ contains the image of $\Gamma_0(n)$ under the composition $\mathrm{tr} \circ \pi$. Let $\mathcal{S}_n$ denote the preimage in $\mathbb{Z}$ of this set under the standard projection $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. We claim that $\mathrm{tr}\,(\Gamma_0(n)) = \mathcal{S}_n$. To see this, let $a \in \{1, \dots, n-1\}$ be coprime to n, and let $d \in \{1, \dots, n-1\}$ be such that $ad \equiv 1 \mod n$. Then $ad = 1 + bn$ for some integer b, and so the matrix

$$\begin{pmatrix} a & b \\ n & d \end{pmatrix} \in \Gamma_0(n)$$

has trace $a + d$. Furthermore, the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m \begin{pmatrix} a & b \\ n & d \end{pmatrix} = \begin{pmatrix} a + mn & b + md \\ n & d \end{pmatrix}$$

ensure that all integers of the form $(a + d) + mn$, $m \in \mathbb{Z}$, appear as traces of elements of $\Gamma_0(n)$.

When $n \geq 5$, we generate the subgroup H_n as follows. Let $\{a_i\} \subset \{1, \dots, n-1\}$ be a complete set of residue classes coprime to n, and for each i, let $d_i \in \{1, \dots, n-1\}$ be such that $a_i d_i \equiv 1 \mod n$; that is, $a_i d_i = 1 + b_i n$ for some $b_i \in \mathbb{Z}$. By Lemma 3.2, since $n \geq 5$, the subgroups

$$G_i = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a_i & b_i \\ n & d_i \end{pmatrix} \right\rangle.$$

are of infinite index in $\Gamma_0(n)$ for each i.

For $n = 2, 3, 4$ the above method does not generate infinite index subgroups, so we treat these cases individually. For $n = 2$, we take the subgroups

$$G_1 = \left\langle \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle, G_2 = \left\langle \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \right\rangle.$$

For $n = 3$, we take the subgroups

$$G_1 = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \right\rangle, G_2 = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix} \right\rangle.$$

For $n = 4$, the subgroup

$$G_1 = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \right\rangle$$

generates all the required traces, and has infinite index in $\Gamma_0(4)$. This treats all cases, and we are done. $\square$

Corollary 3.5. *For p prime, the maximal arithmetic Fuchsian group $N(\Gamma_0(p))$ has infinitely many finite index subgroups with the same trace set at itself.*

Proof. It is known (see Chinburg and Friedman [6] and Maclachlan [10]) that elements of $N(\Gamma_0(p))$ either belong to $\Gamma_0(p)$ or have the form

$$\begin{pmatrix} a\sqrt{p} & \frac{b}{\sqrt{p}} \\ c\sqrt{p} & d\sqrt{p} \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}$ and the determinant is 1. To obtain all rational integer traces, we use the same collection of infinite index subgroups which were used in Corollary 3.4; when $p \geq 5$, we add to this collection the subgroup

$$\left\langle \begin{pmatrix} 0 & \frac{-1}{\sqrt{p}} \\ \sqrt{p} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle,$$

which has a Ford domain of infinite area, and generates all traces of the form $m\sqrt{p}$ for $m \in \mathbb{Z}$. When $p = 2$, we add the two subgroups

$$\left\langle \begin{pmatrix} 0 & \frac{-1}{\sqrt{2}} \\ \sqrt{2} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} \sqrt{2} & \frac{-1}{\sqrt{2}} \\ \sqrt{2} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \right\rangle,$$

and when $p = 3$, we add the subgroups

$$\left\langle \begin{pmatrix} 0 & \frac{-1}{\sqrt{3}} \\ \sqrt{3} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} \sqrt{3} & \frac{-1}{\sqrt{3}} \\ \sqrt{3} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \right\rangle.$$

$\square$

We remark that the next result is closely related to a theorem of Schmutz [13]. In particular, Theorem 3 of [13] gives infinitely many non-isometric surfaces with the same trace set as certain principal congruence subgroups. Our result shows that this holds for every principal congruence subgroup.

Corollary 3.6. *For each $2 \leq n \in \mathbb{N}$, the principal congruence subgroup $\Gamma(n) < \mathrm{PSL}_2(\mathbb{Z})$ admits a finitely generated, infinite index subgroup $H'_n < \Gamma(n)$ such that $\mathrm{tr}\,(H'_n) = \mathrm{tr}\,(\Gamma(n))$.*

Proof. An element of $\Gamma(n)$ can be given the form

$$\begin{pmatrix} 1 + an & bn \\ cn & 1 + dn \end{pmatrix}$$

for integers a, b, c, d. Thus traces of such elements have the form $2 + (a + d)n$. The determinant is

$$1 + (a + d)n + adn^2 - bcn^2 = 1,$$

from which we deduce that $(a + d)n$ is an integer multiple of n^2; this implies that $(a + d)$ is a multiple of n. Hence, the trace set $\mathrm{tr}\,(\Gamma(n))$ contains only elements of the form $2 \pm An^2$, for $A \in \mathbb{Z}$. Moreover, for each integer A, the matrix

$$\begin{pmatrix} 1 & An \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = \begin{pmatrix} 1 + An^2 & An \\ n & 1 \end{pmatrix}$$

realizes this trace. Thus $\mathrm{tr}\,(\Gamma(n)) = \{An^2 \pm 2 \mid A \in \mathbb{N}_0\}$. These traces can all be obtained from the subgroup

$$\left\langle \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \right\rangle,$$

and by Lemma 3.2, we find that when $n > 2$, this subgroup is of infinite index. We take H'_n to be this subgroup. When $n = 2$, we take

$$H'_2 = \left\langle \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle;$$

this subgroup has the required trace set and is of infinite index. □

These results also raise interesting questions. One defines a group commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ to be a *congruence subgroup* if it contains a principal congruence subgroup $\Gamma(n)$ for some n. In the light of the above results, it is natural to ask:

Question. Does every congruence subgroup commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ admit an infinite index subgroup with the same trace set as itself?

The above results also rely on the congruences, and related trace information, which are induced by being a congruence subgroup. Since there exist Fuchsian groups commensurable with $\mathrm{PSL}_2(\mathbb{Z})$ which are not congruence subgroups, it is also pertinent to ask whether there exist non-congruence subgroups with this property.

Question. What can we say about non-congruence subgroups commensurable with $\mathrm{PSL}_2(\mathbb{Z})$?

4 Biancchi Groups

In this section, we show that there are results analogous to Corollary 1.2 for each Bianchi group. As above, there is a more general result involving any trace set that can be written as the union of finitely many (translates of) sublattices of the ring of integers. For brevity, we will only prove an analogue of Corollary 1.2 for each Bianchi group, rather than an analogue of the more general Theorem 1.1.

Consider the Bianchi group $\mathrm{PSL}_2(\mathcal{O}_d)$, where $d > 0$ is a square-free integer, $\mathcal{O}_d$ denotes the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. It is a standard fact that $\mathcal{O}_d$ is an integer lattice in $\mathbb{C}$ generated by 1 and $\omega = \sqrt{-d}$ (if $d \equiv 1, 2 \bmod 4$) or by 1 and $\omega = \frac{1+\sqrt{-d}}{2}$ (if $d \equiv 3 \bmod 4$).

Theorem 1.3. *Given any Bianchi group* $\mathrm{PSL}_2(\mathcal{O}_d)$*, there are infinitely many finite index subgroups* $\Gamma <$ $\mathrm{PSL}_2(\mathcal{O}_d)$ *with the same complex trace set as* $\mathrm{PSL}_2(\mathcal{O}_d)$.

Proof. As in the Fuchsian case, we construct finitely many subgroups which together generate every trace, and then conjugate them so that together they generate a subgroup of infinite index. We then appeal to the fact that the Bianchi groups are LERF. The following general method works whenever there are no non-real integers in $\mathcal{O}_d$ of complex modulus less than 2; this includes all Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_d)$ for $d \neq 1, 2, 3$; we deal with these cases afterwards.

The subgroups which we take are the five subgroups of the form

$$P_x = \left\langle \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 3\omega \\ 0 & 1 \end{pmatrix} \right\rangle,$$

for $x \in \{0, 1, \omega, 1 + \omega, 2 + \omega\}$. In each case, a Ford domain for P_i is bounded by isometric spheres of radius 1 centered at 0 and x respectively, together with vertical planes which form the boundary of a fundamental domain for the parabolics fixing ∞. We conjugate the P_i so that we may apply Theorem 2.1; to do this, we conjugate by involutions δ_i which have isometric spheres disjoint from those of the P_i and from each other. If $d \equiv 1, 2 \bmod 4$, we take

$$\delta_1 = \begin{pmatrix} 38+85\omega & -17-76\omega-85\omega^2 \\ 85 & -38-85\omega \end{pmatrix} = \begin{pmatrix} 38+85\omega & 85d-17-76\omega \\ 85 & -38-85\omega \end{pmatrix},$$

$$\delta_\omega = \begin{pmatrix} 7 & -10 \\ 5 & -7 \end{pmatrix}, \delta_{1+\omega} = \begin{pmatrix} 68 & -125 \\ 37 & -68 \end{pmatrix}, \delta_{2+\omega} = \begin{pmatrix} 43 & -50 \\ 37 & -43 \end{pmatrix};$$

and if $d \equiv 3 \bmod 4$, we take

$$\delta_1 = \begin{pmatrix} -2+3\omega+4d\omega & 1+4\omega-7\omega^2-4\omega^2 d \\ 4d-1 & 2-3\omega-4d\omega \end{pmatrix},$$

$$\delta_\omega = \begin{pmatrix} 7 & -10 \\ 5 & -7 \end{pmatrix}, \delta_{1+\omega} = \begin{pmatrix} 7-5\omega & 5d-10+14\omega \\ 5 & 5\omega-7 \end{pmatrix}, \delta_{2+\omega} = \begin{pmatrix} 43 & -50 \\ 37 & -43 \end{pmatrix}.$$

For the three remaining cases, we choose conjugations specific to each case, and take care because isometric spheres of higher powers of the generators of the subgroups may appear. We treat the cases $d = 1, 2$ together. We take the same P_i as above, and conjugations

$$\delta_1 = \begin{pmatrix} 38+85\omega & 85d-17-76\omega \\ 85 & -38-85\omega \end{pmatrix}, \delta_\omega = \begin{pmatrix} 43 & -50 \\ 37 & -43 \end{pmatrix},$$

$$\delta_{1+\omega} = \begin{pmatrix} 68 & -125 \\ 37 & -68 \end{pmatrix}, \delta_{2+\omega} = \begin{pmatrix} 91 & -101 \\ 82 & -91 \end{pmatrix}.$$

The last case is where $d = 3$. We take

$$\delta_1 = \begin{pmatrix} -2+3\omega+4d\omega & 1+4\omega-7\omega^2-4\omega^2 d \\ 4d-1 & 2-3\omega-4d\omega \end{pmatrix}, \delta_\omega = \begin{pmatrix} 43 & -50 \\ 37 & -43 \end{pmatrix},$$

$$\delta_{1+\omega} = \begin{pmatrix} 68-37\omega & 99\omega-88 \\ 37 & 37\omega-68 \end{pmatrix}, \delta_{2+\omega} = \begin{pmatrix} 68 & -125 \\ 37 & -68 \end{pmatrix}.$$

By construction, the subgroups P_0 and $\delta_x P_x \delta_x$ for $x \in \{1, \omega, 1+\omega, 2+\omega\}$ satisfy the hypotheses of Theorem 2.1, and so the group H generated by P_0 and the $\delta_x P_x \delta_x$ is the free product of the generating subgroups, has infinite index in $\mathrm{PSL}_2(\mathcal{O}_d)$, and has all the same traces as $\mathrm{PSL}_2(\mathcal{O}_d)$. Since the Bianchi group is LERF, this implies that there exists a proper finite index subgroup $K_d < \mathrm{PSL}_2(\mathcal{O}_d)$ which contains this subgroup, but not

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and therefore is a proper, finite index subgroup with the same trace set as the Bianchi group. Moreover, by the alternative formulation of LERF, H can be written as the intersection of infinitely many finite index subgroups of $\mathrm{PSL}_2(\mathcal{O}_d)$, each of which therefore has trace set the same as $\mathrm{PSL}_2(\mathcal{O}_d)$. $\square$

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