

Matrix Realizations of the Hyperbolic Tetrahedral Groups in $\mathrm{PSL}_2(\mathbb{C})$

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The purpose of this note is the construction of matrices in $\mathrm{PSL}_2(\mathbb{C})$ generating each of the 32 hyperbolic tetrahedral groups. Explicit generators for some of these groups are given in [1], [2] and [3]; below we give a method which yields such generators for all of the cases.

Non-Compact Tetrahedra

Given one of the 23 non-compact hyperbolic tetrahedra $T(\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3)$ listed in [1], we place it in the upper half-space model of \mathbb{H}^3 by putting an ideal vertex at ∞ and the other three vertices on the unit sphere. Then the three edges labelled μ_1, μ_2 and μ_3 are vertical and meet at ∞ , and the three edges labelled λ_1, λ_2 and λ_3 lie on the unit sphere. The plane P defined by

$$y = -\cos \frac{\pi}{\lambda_3}$$

intersects the unit sphere with dihedral angle $\frac{\pi}{\lambda_3}$. Define two complex numbers as follows:

$$q_1 = \frac{-\cos \frac{\pi}{\lambda_2} - \cos \frac{\pi}{\lambda_3} \cos \frac{\pi}{\mu_1}}{\sin \frac{\pi}{\mu_1}} - i \cos \frac{\pi}{\lambda_3};$$

and

$$q_2 = \frac{\cos \frac{\pi}{\lambda_1} + \cos \frac{\pi}{\lambda_3} \cos \frac{\pi}{\mu_2}}{\sin \frac{\pi}{\mu_2}} - i \cos \frac{\pi}{\lambda_3}.$$

The hyperbolic geodesics above these points lie in the plane P , defining a face of the tetrahedron. The following matrices represent rotations of the appropriate order around each respective edge.

$$A_1 = \begin{pmatrix} e^{\frac{i\pi}{\mu_1}} & q_1(e^{\frac{-i\pi}{\mu_1}} - e^{\frac{i\pi}{\mu_1}}) \\ 0 & e^{\frac{-i\pi}{\mu_1}} \end{pmatrix}$$

is rotation of order μ_1 around the axis above the point q_1 ;

$$A_2 = \begin{pmatrix} e^{\frac{i\pi}{\mu_2}} & q_2(e^{\frac{-i\pi}{\mu_2}} - e^{\frac{i\pi}{\mu_2}}) \\ 0 & e^{\frac{-i\pi}{\mu_2}} \end{pmatrix}$$

is rotation of order μ_2 around the axis above the point q_2 ; and

$$A_3 = \begin{pmatrix} 2 \cos \frac{\pi}{\lambda_3} & i \\ i & 0 \end{pmatrix}$$

is rotation of order λ_3 around the intersection of P and the unit sphere. One now checks - for example, by examining traces - that these matrices satisfy the additional relations required in the tetrahedral groups, and that they therefore realize the presentation

$$\{A_1, A_2, A_3 \mid A_1^{\mu_1} = A_2^{\mu_2} = A_3^{\lambda_3} = (A_3 A_1)^{\lambda_2} = (A_3 A_2^{-1})^{\lambda_1} = (A_1 A_2)^{\mu_3} = 1\}.$$

Compact Tetrahedra

Again we will place the given tetrahedron in upper half-space such that one face lies on the unit sphere, and two of the other three faces lie in vertical planes. To simplify the calculations, we will exploit the fact that in all cases, at least one dihedral angle is equal to $\frac{\pi}{2}$, and so we may rearrange the parameters such that $\lambda_3 = 2$. Thus, $\cos \frac{\pi}{\lambda_3} = 0$, and the plane P is defined by $y = 0$, so is the x - z plane. The point q_1 now has the simpler form

$$q_1 = \frac{-\cos \lambda_2}{\sin \mu_1}$$

and, as above, we have one edge on the unit sphere and another above the point q_1 . It thus remains to identify the circle C in the x - z plane which contains the third edge of the face P . Say the extreme points of this circle - that is, the endpoints of the hyperbolic geodesic - are $a, b \in \mathbb{R}$ such that $b < a$. Note that we necessarily have $b < q_1 < a$, that the center of the circle is at $\frac{a+b}{2}$, and that the radius of the circle is $\frac{a-b}{2}$. Values for a and b can be determined as follows. The face angles at the vertex $v_1 = (q_1, 0, z_1)$, where C intersects the line $x = q_1$, and the vertex $v_2 = (x_2, 0, z_2)$, where C intersects the line $x^2 + z^2 = 1$ on the unit sphere, can be found from the dihedral angles by using the spherical cosine law

$$\cos \alpha_i = \frac{\cos \beta_i + \cos \beta_j \cos \beta_k}{\sin \beta_j \sin \beta_k}$$

where α_i is the required face angle, β_i the dihedral angle at the edge not contained in the face, and β_j, β_k the dihedral angles at the edges contained in the face. We thus have a Euclidean right triangle with its vertices $\frac{a+b}{2}, q_1$ and v_1 . This has face angle θ_1 at the center of C , where

$$\cos \theta_1 = \frac{\cos \mu_3 + \cos \mu_1 \cos \mu_2}{\sin \mu_1 \sin \mu_2},$$

and so the right triangle gives the equation

$$\cos \theta_1 = \frac{2q_1 - (a + b)}{a - b}. \tag{1}$$

The Euclidean triangle with vertices at $\frac{a+b}{2}, 0$ and v_2 has angle θ_2 at v_2 , where

$$\cos \theta_2 = \frac{\cos \lambda_1 + \cos \mu_2 \cos \lambda_3}{\sin \mu_2 \sin \lambda_3} = \frac{\cos \lambda_1}{\sin \mu_2},$$

side length $\frac{a+b}{2}$ opposite θ_2 , and side lengths 1 and $\frac{a-b}{2}$ adjacent to θ_2 . The cosine law then gives

$$\cos \theta_2 = \frac{1 - ab}{a - b}. \quad (2)$$

We hence seek solutions (a, b) to the equations (1) and (2) with the constraints given above. Solving (1) for a , we find

$$a = \frac{2q_1 + b(\cos \theta_1 - 1)}{\cos \theta_1 + 1},$$

and now (2) becomes the quadratic

$$b^2(\cos \theta_1 - 1) + b(2q_1 - 2\cos \theta_2) + 2q_1 \cos \theta_2 - \cos \theta_1 - 1 = 0.$$

Fixing b to be the lesser root gives a and b satisfying the required constraints, and thus determines the circle C . The generators of the tetrahedral group are now

$$A_1 = \begin{pmatrix} e^{\frac{i\pi}{\mu_1}} & q_1(e^{\frac{-i\pi}{\mu_1}} - e^{\frac{i\pi}{\mu_1}}) \\ 0 & e^{\frac{-i\pi}{\mu_1}} \end{pmatrix},$$

$$A_2 = \frac{1}{b-a} \begin{pmatrix} b & a \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{\frac{i\pi}{\mu_2}} & 0 \\ 0 & e^{\frac{-i\pi}{\mu_2}} \end{pmatrix} \begin{pmatrix} 1 & -a \\ -1 & b \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

These matrices satisfy the relations

$$\{A_1, A_2, A_3 \mid A_1^{\mu_1} = A_2^{\mu_2} = A_3^{\lambda_3} = (A_3 A_1)^{\lambda_2} = (A_3 A_2)^{\lambda_1} = (A_2 A_1)^{\mu_3} = 1\}.$$

References

- [1] Andrew M. Brunner, Youn W. Lee, and Norbert J. Wielenberg, *Polyhedral groups and graph amalgamation products*, *Topology Appl.* **20** (1985), no. 3, 289–304. MR 804041 (87a:57049)
- [2] Tobias Finis, Fritz Grunewald, and Paulo Tirao, *The cohomology of lattices in $SL(2, \mathbb{C})$* , *Experiment. Math.* **19** (2010), no. 1, 29–63. MR 2649984
- [3] Colin Maclachlan and Alan W. Reid, *The arithmetic of hyperbolic 3-manifolds*, *Graduate Texts in Mathematics*, vol. 219, Springer-Verlag, New York, 2003. MR 1937957 (2004i:57021)