MAT 3701: Selected Solutions to Assignment 2

February 11, 2013

1 Exercises from the text

2.6 The set A is the disjoint union of $A \cap B$ and $A \cap B'$. Therefore, by the additive property of probability measure, $P(A) = P(A \cap B) + P(A \cap B')$. Subtracting $P(A \cap B)$ from both sides yields the desired result: $P(A \cap B') = P(A) - P(A \cap B)$.

2 Additional Exercises

Explain all answers completely in essay form!

1. Given integers n and k, suppose we wish to choose n integers whose sum is less than or equal to k, that is, integers $i_1, i_2, i_3, \ldots, i_n$ such that $\sum_{j=1}^{n} i_j \leq k$. Show that the number of ways this can be done is $\binom{n+k}{n}$. Note that the integers $\{i_j\}_{j=1}^{n}$ do not have to be distinct. (Big hint: Imagine a row of $n + k$ tokens (sticks, pennies, or whatever). Choose n of these to remove, and let the numbers $i_1, i_2, i_3, \ldots, i_n$ be the number of tokens in the first n groups separated by the those that have been removed. There may be a non-empty $(n + 1)^{st}$ group of leftovers, in which case the sum is strictly less than k.)

   Think of choosing these n integers whose some is less than or equal to k as distributing k tokens among $n+1$ receptacles. The last ($(n+1)^{\text{st}}$) receptacle contains the remainder when the sum of these integers is subtracted from k (since the total can be less); the first n receptacles contain the integers we wish to choose. Thus, we can view this as an occupancy problem. We know from previous exercises that the number of ways to carry out the process above is $\binom{k+n}{n} = \binom{k+n}{k}$. (We would use whichever of these binomial coefficients is more convenient to do the calculation if we needed to.)

   Remark. Do not forget that, for the purpose of probability calculations, if the tokens are distributed at random the occupancy outcomes are *not* equally likely.

2. Given integers n and k, suppose we wish to choose n integers whose sum is *exactly* equal to k, that is, integers $i_1, i_2, i_3, \ldots, i_n$ such that $\sum_{j=1}^{n} i_j = k$. Show that the number of ways this can be done is $\binom{n+k-1}{n-1}$. (Hint: In this case, we don't allow any leftovers.) In addition, explain how this calculation is related to occupancy problems.)

Simply modify the process above by eliminating the last receptacle, since there will be no remainder if the sum is exactly k. Distributing the tokens amounts to dividing the desired total among the n terms.

3. Please excuse the typo in the original; as we know, the sum should alternate, as in the revised formula below.

Given events $E_1, E_2, E_3, \ldots, E_n$ from a sample space S with probability measure ρ, let $S_k = \sum_{1 \le i_1 < i_2 < i_3 < \cdots < i_k \le 1} E_{i_1} \cap E_{i_2} \cap E_{i_3} \cdots \cap E_{i_k}$. Prove that $\rho(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) = \sum_{k=1}^{n} (-1)^{k-1} S_k$.

Proof. The proof proceeds, of course, by induction. For the case $n = 2$ the result has been proven in class and in the text using the additivity property of probability measures. This takes care of the initial case. For the general case we must prove, for all integers n, the conditional proposition that if $\rho(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_{n-1}) = \sum_{k=1}^{n-1} (-1)^{k-1} S_k$, then $\rho(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) = \sum_{k=1}^{n} (-1)^{k-1} S_k$ Thus, assume as inductive hypothesis that $\rho(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_{n-1}) = \sum_{k=1}^{n-1} (-1)^{k-1} S_k$ and consider $\rho(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n)$. In what follows it is important to remember that the inductive hypothesis applies to *any* collection of $n-1$ sets; each E_i is just a universally quantified variable. Note also the notational sloppiness, which would be cumbersome to fix: each S_k depends on the collection of sets to which it is being applied; S_k is not a constant. In reading what follows you must pay careful attention to which collection of sets is being considered. To help you I will henceforth use S'_k when only the sets $E_1, E_2, E_3, \ldots, E_{n-1}$ are being considered and S''_k when the sets $E_1 \cap E_n, E_2 \cap E_n, E_3 \cap E_n, \ldots, E_{n-1} \cap E_n$ are being considered. (You will see in due course how this last collection arises.)

The operation of union is associative, so $E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n = (E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_{n-1}) \cup E_n$. Thus, by the inductive hypothesis, the initial case of two sets already proven, and a little basic set theory we have

$$\rho(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_n) = \rho(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_{n-1}) + \rho(E_n) - \rho(E_1 \cup E_2 \cup E_3 \cup \cdots \cup E_{n-1}) \cap E_n$$

$$= \sum_{k=1}^{n-1} (-1)^{k-1} S'_k + \rho(E_n) - \rho((E_1 \cap E_n) \cup (E_2 \cap E_n) \cap (E_3 \cap E_n) \cap \cdots \cap (E_{n-1} \cap E_n)).$$

Now the trick is to recognize how the second and third terms in this formula relate to the various sums S_k applied to *all* of the sets, E_n included. First consider $\rho(E_n)$: when added to $S'_1 = \sum_{k=1}^{n-1} \rho(E_k)$, as applied to $E_1, E_2, E_3, \ldots, E_{n-1}$, it gives us $S_1 = \sum_{k=1}^{n} \rho(E_k)$, as applied to $E_1, E_2, E_3, \ldots, E_n$. Next consider $\rho((E_1 \cap E_n) \cup (E_2 \cap E_n) \cap (E_3 \cap E_n) \cap \cdots \cap (E_{n-1} \cap E_n))$: there are only $n-1$ sets in this union, so the inductive hypothesis applies. Thus we have

$$\rho((E_1 \cap E_n) \cup (E_2 \cap E_n) \cap (E_3 \cap E_n) \cap \cdots \cap (E_{n-1} \cap E_n)) = \sum_{k=1}^{n-1} (-1)^{k-1} S''_k,$$

where

$$S''_k = \sum_{1 \le i_1 < i_2 < i_3 < \cdots < i_k \le 1} (E_{i_1} \cap E_n) \cap (E_{i_2} \cap E_n) \cap (E_{i_3} \cap E_n) \cdots \cap (E_{i_k} \cap E_n)$$
$$= \sum_{1 \le i_1 < i_2 < i_3 < \cdots < i_k \le 1} E_{i_1} \cap E_{i_2} \cap E_{i_3} \cap \cdots \cap E_{i_k} \cap E_n.$$

Finally, it is only necessary to note at this point that the terms in each S''_k are exactly the ones that must be added to S'_{k+1} to obtain S_{k+1}, namely the intersections that involve E_n. (The offset in the index results from the fact that the intersections in S''_k involve $k+1$ sets, since E_n is included along with $E_{i_1}, E_{i_2}, E_{i_3}, \ldots, E_{i_k}$.) To fit this pattern and summarize the final steps succinctly, set $S''_0 = \rho(E_n)$ and $S'_n = 0$. Then $S_k = S'_k + S''_{k-1}$, for $k = 1, 2, 3, \ldots, n$, and we have

$$\rho(\bigcup_{i=1}^{n} E_i) = \sum_{k=1}^{n-1} (-1)^{k-1} S'_k + \rho(E_n) - \rho((E_1 \cap E_n) \cup (E_2 \cap E_n) \cap (E_3 \cap E_n) \cap \cdots \cap (E_{n-1} \cap E_n))$$

$$= \sum_{k=1}^{n-1} (-1)^{k-1} S'_k + S''_0 - \sum_{k=1}^{n-1} (-1)^{k-1} S''_k = \sum_{k=1}^{n} (-1)^{k-1} S'_k + \sum_{k=0}^{n-1} (-1)^{k} S''_k = \sum_{k=1}^{n} (-1)^{k-1} [S'_k + S''_{k-1}]$$
$$= \sum_{k=1}^{n} (-1)^{k-1} S_k.$$

□

I recognize that this proof is a very tough exercise. Also a good one to study and learn from. What can you learn? Among other things, to be patient and persistent and systematic: there is no great flash of insight needed for this proof, just careful observation and attention to detail. Learn as well to revise and refine your notation and arguments as needed until the proof is complete and clear. To proceed strategically, step by step. To have confidence that if you proceed in this way you can succeed, and to experience real satisfaction when you do. Mathematics is hard and it takes time, but it has its rewards!

4. If a (fair) coin is tossed n times, what is the probability that a particular given sequence of heads and tails will be tossed? (Hint: Each possible sequence is equally likely. How many are there?)

There are 2^n such sequences, each equally likely. Thus each sequence has probability $\frac{1}{2^n}$.

5. If a coin is tossed three times, what is the probability that it comes up heads two out of the three times?

Of the $2^3 = 8$ equally likely outcomes, there are three sequences with two heads and one tail. (There are three positions in which the single tail can come up.) Thus the probability of this event is $\frac{3}{8}$.

6. Suppose a coin is tossed n times. What is the probability that it comes up heads at least once?

For this one we observe from experience that it is easier to find the probability of the complementary event and subtract. The probability of getting all tails, since there is only one way to do that, is $\frac{1}{2^n}$; thus, the probability that at least one head will be obtained is $1 - \frac{1}{2^n}$.

7. An urn contains 5 red balls, 2 black balls, and 3 white balls. (Probability theorists are inordinately fond of urns, the idea presumably being that one cannot see into them.) Three balls are picked at random.

The sample space consists of all the ways that 3 balls can be chosen from the 10 balls in the urn. (Note that the problem assumes that all the balls are picked at once, thus without replacement.) It follows that the sample space contains $\binom{10}{3}$, all of which are, of course, equally likely (since nothing favors one ball over another).

(a) What is the probability that one ball of each color is picked? This event contains $\binom{5}{1}\binom{2}{1}\binom{3}{1}$ outcomes. This calculation is made by noting that we can count by first choosing a red ball, then a black ball, and then a white ball. Thus the probability of this event is $\frac{\binom{5}{1}\binom{2}{1}\binom{3}{1}}{\binom{10}{3}}$.

(b) What is the probability that three red balls are picked? This event contains $\binom{5}{3}$ outcomes. Thus the probability of this event is $\frac{\binom{5}{3}}{\binom{10}{3}}$.

(c) What is the probability that no red balls are picked? This event contains $\binom{5}{3}$ outcomes, since there are five balls that are not red. Thus the probability of this event is $\frac{\binom{5}{3}}{\binom{10}{3}}$.

(d) What is the probability that two black balls are among those picked? The black balls must both be picked, and there is only one way to do that. The other ball can be any among the remaining eight. Thus, this event contains $\binom{8}{1} = 8$ outcomes. Thus the probability of this event is $\frac{8}{\binom{10}{3}}$.

(e) What is the probability that two black balls and one white ball are picked? Again, both black balls must be picked, and there is only one way to do that. The remaining ball must be from among the white balls, so this event contains $\binom{3}{1} = 3$ outcomes. Thus the probability of this event is $\dfrac{3}{\binom{10}{3}}$.

(f) If you know that one of the balls is black, what is the probability that the other two are red? Knowing that one ball is black amounts to restricting the sample space to the event that one ball is black. There are $\binom{2}{1}\binom{5}{2}$ ways to pick one black ball and two red balls, and there are $\binom{2}{1}\binom{9}{2}$ ways just to include a black ball among those picked. Thus the probability that the other two balls are red, given that one is black, is $\frac{\binom{2}{1}\binom{5}{2}}{\binom{2}{1}\binom{9}{2}} = \frac{\binom{5}{2}}{\binom{9}{2}}$.

Alternatively, we can use the definition of conditional probability: $P(E|F) = \frac{P(E \cap F)}{P(F)}$. $P(E \cap F) = \frac{\binom{2}{1}\binom{5}{2}}{\binom{10}{2}}$, and $P(F) = \frac{\binom{2}{1}\binom{9}{2}}{\binom{10}{2}}$. It is instructive to notice that the denominators just cancel, and of course we obtain the same answer as above.

As yet another approach, imagine that we simply start by picking one of the black balls. We can then reduce our sample space to the possible outcomes for the remaining two balls. These two are picked from the nine balls remaining in the urn, of which one is black, five red, and three are white. Thus, in this case the sample space must be restricted to $\binom{9}{2}$ outcomes, of which $\binom{5}{2}$ are in the event in question. Thus again we find that the probability of this event is $\dfrac{\binom{5}{2}}{\binom{9}{2}}$.

9. Suppose a spinner is constructed that is equally likely to stop at any point of the compass when spun. What is the probability that it will land between 0° (North) and 22°?

Since each point of the compass is equally likely, the probability of any event is proportional to the central angle it spans. There are 360° in the full circle, whereas the event in question spans 22°. So it probability is $\frac{22}{360} = \frac{11}{180}$.

10. If we know the spinner landed between 0° (North) and 22° but don't know exactly where it landed, what is the probability that it lies between 5° (North) and 10°?

In this situation the given condition restricts our sample space to a span of 22°, whereas the event in question spans 5°. Thus its probability is $\frac{5}{22}$.