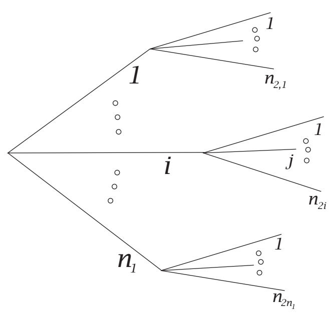
MAT 3701 (Spring 2013): Selected solutions to Problem Set 1

January 24, 2013

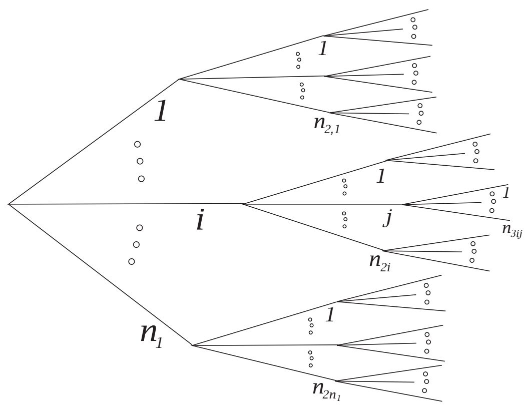


1.1 (a) If the first step is carried out in the i^{th} way, there are n_{2i} ways to carry out the second step; thus, there are n_{2i} combinations that begin with the i^{th} way of doing the first step. (Visualize them as paths from the root to the tip of the tree.) This number may vary with i, so we must simply add up all the combinations:

$$\sum_{i=1}^{n_1} n_{2i}$$

Notational remark: When using double subscripts, a comma is necessary between numbers so that they do not look like digits of the same number; no comma is needed between letters or between letters and numbers.

(b) In this example $n_1 = 4$, but it is most natural to let i vary from 0 to 3 instead of from 1 to 4; then $n_{2,0} = n_{2,1} = 4$, $n_{2,2} = 3$, and $n_{2,3} = 2$. Thus there are indeed $4 + 4 + 3 + 2 = 13$ ways the student can study.



1.3 (a)

If the first step is carried out in the i^{th} way *and* the second step is carried out in the j^{th} way, then there are n_{3ij} ways in which the thirds step may be done. Thus there are n_{3ij} combinations that begin with the i^{th} way of doing the first step followed by the j^{th} way of doing the second step. Since this number may vary with i and j, we must add up the combinations over all possible values of i and j:

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_{2i}} n_{3ij}$$

Notice that the sum over i must be on the outside, since the domain for j varies with i.

1.8 Think of the operation as choosing a cell (or box, urn, etc.) for each object. Since cells may be used repeatedly (that is, any number of objects may be placed in a cell), there are always n choices for each object, hence we can multiply to obtain n^r ways to distribute the objects.

1.9 If the objects are indistinguishable an entirely different method is needed; all that we can distinguish is the number of objects in each cell. Think of the operation as lining the objects up together with enough dividers to separate the objects into a group for each cell. Thus, the objects to the left of the first divider will go to the first cell, those between the first two dividers will go to the second cell, and so forth, with the objects to the right of the last divider going into the last cell. Since there are n cells, $n-1$ dividers are required. The positions of the dividers uniquely determine the distribution, so we need only determine the number of ways these positions can be chosen from the total of $r+n-1$ positions. As we know, there are $\binom{r+n-1}{n-1} = \binom{r+n-1}{r}$ ways to do this.

1.10 Think of the operation as having two steps: first put one object in each cell, in order to ensure that each cell has at least one; the remaining objects are distributed at random. Hence the problem reduces to the number of ways that $r - n$ objects can occupy n cells: $\binom{r-n+n-1}{r-n} = \binom{r-1}{r-n}$. (Obviously it is necessary that $n \leq r$.)

1.14 (a) Let $x = y = 1$.

(b) Let $x = 1$ and $y = -1$.

(c) Let $x = 1$ and $y = a - 1$.

1.16 Let $m = k = n$. Don't forget the symmetry of binomial coefficients: $\binom{n}{r} = \binom{n}{n-r}$. (Choosing r objects to take is the same as choosing $n - r$ objects to leave behind.)

1.23 Rather than the algebraic method suggested by the text, I would take a more logical approach. We wish to partition n objects into k subsets of specified size $n_1, n_2, n_3, \ldots, n_k$. Fix a particular object. There are k distinct possibilities for where this object ends up. If it is in the i^{th} subset, then we already know where it is, and there are $\binom{n-1}{n_1, n_2, \ldots, n_i - 1, \ldots, n_k}$ ways to partition the remaining objects. Thus in total there are $\sum_{i=1}^{k} \binom{n-1}{n_1, n_2, \ldots, n_i - 1, \ldots, n_k}$ ways to partition the objects, which verifies the result. (Much easier than doing all that algebra, no?!)