# MAT 3701: Mathematical Expectation

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Mathematical expectation is a means of placing a value on a random variable. Suppose for example that I offer you the following wager: I will toss a fair coin, and if it comes up heads I will pay you two dollars, whereas if it comes up tails I will pay you nothing. Naturally, I expect you to put down some money to play. How much would you be willing to put down, assuming you want to come out ahead? (Or, from a different point of view, what is the minimum bet I would accept, assuming *I* want to come out ahead?) Given that you have a 50% chance of getting nothing and a 50% chance of winning \$2.00, it is reasonable to *expect*, on average, to win \$1.00: $(.5)(\$2.00) + (.5)(\$0.00) = \$1.00$. It can be shown, in fact, that if you play the game many times, your average winnings per game will be increasingly likely to be close to \$1.00. Thus, if you want to come out ahead, you should bet less than \$1.00. (And if I want to come out ahead, I must insist that you bet more than \$1.00. Thus, if we both want to come out ahead, the game will not be played! It will only be played if one party is willing to gamble on "beating the odds," which no rational person would do, but some people get a thrill out of taking a chance.)

Expectation is a sort of weighted average, just like your final grade in this class. We are simply weighing each possible outcome by its likelihood of occurring. In general, an average value need not be equal to any of the actual values being averaged. If there are two people, one of whom is 6 feet tall and one of whom is 5 feet tall, their average height is $5\frac{1}{2}$ feet; neither person actually realizes the average value. Similarly, you will never actually win \$1.00 on any single play of the game just described, as it is only possible to win \$0.00 or \$2.00.

The concept of expectation is a simple one, but as often happens in mathematics, there are subtleties involved when there are infinitely many possibilities. Consider, for example, the following game of chance: I start by putting a dollar into the "pot," and then I repeatedly flip a fair coin. Every time a head comes up, I double the pot. The first time a tail comes up, the game is over, and you get whatever is in the pot. Before we analyze the game mathematically, consider on the basis of your intuition how much you would be willing to pay to play this game.

To analyze the game, let $Y$ be the amount in the pot at the end of the game. You should recognize $Y$ as a function of a geometrically distributed random variable, let's call it $X$, representing the number of the toss on which a tail first appears. If a tail comes up on toss $x$, then you win $y = 2^{x-1}$ dollars. The probability that $X = x$ is, as we know, $\frac{1}{2^x}$. It seems reasonable, then, to calculate the expectation of $Y$, commonly denoted by $\mathrm{E}Y$, as

$$\mathrm{E}Y = \sum_{i=1}^{\infty} 2^{i-1} \cdot \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty.$$

Your expected winnings are infinite! Nonetheless, I suspect that after considering the pots available in the early stages of the game, you would not be willing to pay very much to play, and neither would most people. This situation, known as the St. Petersburg Paradox, has interested mathematicians, physicists, and economists since Nicolas Bernoulli invented it in 1713 [1]. There are various explanations for why it does not make sense to bet much on this game. The simplest points out that in reality my funds are limited. Even if you are lucky enough to win one of the unlikely but enormous pots that raise the expectation without bound, you won't get all of your money, because you will exhaust my bank account. I'll pay everything I have, but then I'll be bankrupt. It turns out that even if I am enormously rich, your expected earnings will not be very great if the payout is limited. (Think about it: once I max out, the terms in the expectation are successively cut in half. Even if I max out at, say, $2^{1000}$ dollars, which is many, many times more than the combined wealth of all the economies on earth, your expected earnings will be just $\$500.00 + \$.25 + \$.125 + \$.0625 + \cdots = \$500.50$.) Another explanation takes time into consideration, pointing out that even in the hypothetical situation that my funds are unlimited, the game has to go on a long time for the pot to get really large; therefore, the *rate* of return on a large investment is small. Leaving details aside, the upshot of all this is that expectation is clearly not an adequate valuation for all random variables.

To avoid paradoxes of this sort, we confine our attention to random variables that have finite expectation. But in order to have a good definition of what it means for expectation to be finite, we must consider the additional subtleties that arise if a random variable can take negative values. Suppose for example that we modify our game as follows: instead of starting with \$1.00 and doubling the pot for each head, which yielded a pot of $2^{x-1}$ dollars, the pot will be equal to $(-1)^{x-1} \cdot \frac{2^x}{x}$ dollars, where $x$ is the toss on which the first tail appears. In other words, if no heads come up I will pay you 2 dollars, if one head comes up, you will pay me 2 dollars, if two heads come up, I will pay you $\frac{8}{3}$ dollars, and so forth. Let $Z$ denote the amount you win in this new game. Now we might attempt to calculate your expected winnings as

$$\mathrm{E}Z = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2^i}{i} \cdot \frac{1}{2^i} = \sum_{i=1}^{\infty} (-1)^{i-1} \cdot \frac{1}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

You may recognize this series from calculus as the alternating harmonic series. The problem with this series is that there is no reason to add the weighted outcomes in any particular

[1] The paradox is named after a presentation and proposed solution of the problem published by Nicolas' cousin Daniel Bernoulli in the *Commentaries of the Imperial Academy of Science of Saint Petersburg* in 1738. Bernoulli's solution was based on the diminishing marginal utility of money: the more you win, the less each additional dollar means to you.

order, and you get different sums from different orders. For example, if you sum the series in the order given above, the sum is clearly less than one, but if you sum it in the order $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \frac{1}{2} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} - \frac{1}{3} + \cdots$, the sum is greater than one. In other words, if we postulate that addition should be commutative, even for an infinite sum, then this sum is not well-defined. Naturally the expectation of a random variable, if it exists, must be a well-defined quantity; furthermore, in order to prove useful theorems about expectation, it is absolutely necessary to be able to rearrange the terms in the sum.

It is a basic fact of analysis that the sum of a series is well-defined with respect to the order of the terms if and only if the series converges absolutely. (We will not prove this fact here, but you should know it, and the intuition behind it is not difficult: in order for a series to converge absolutely, the terms at the tail must add up to arbitrarily small amounts if you start far enough out, so changing their position doesn't affect the result. It is proved in classes in analysis, such as MAT 4860.) Thus we arrive at the following definition:

**Definition.** Let $X$ be a discrete random variable taking values $x_i$ (where $i$ may range over either a finite or countably infinite index set) and having probability distribution $f$. If $\sum_i |x_i| f(x_i) < \infty$, then $X$ has *finite expectation*, and the *expectation of* $X$ is defined to be $\mathrm{E}X = \sum_i x_i f(x_i)$. If $\sum_i |x_i| f(x_i) = \infty$, then $X$ does not have finite expectation, and $\mathrm{E}X$ is undefined.

We make an analogous definition for continuous random variables, and a similar definition applies to all non-discrete random variables, although understanding the general concept of integration that is needed requires graduate-level measure theory.

**Definition.** Let $X$ be a continuous random variable with probability density $f$. If $\int_{-\infty}^{\infty} |x| f(x) dx < \infty$, then $X$ has *finite expectation*, and the *expectation of* $X$ is defined to be $\mathrm{E}X = \int_{-\infty}^{\infty} x f(x) dx$. If $\int_{-\infty}^{\infty} |x| f(x) dx = \infty$, then $X$ does not have finite expectation, and $\mathrm{E}X$ is undefined.

Using these definitions, we will prove a variety of properties of expectation that make it fairly straightforward to compute.