4.6. Conditionals and Validity

1. Conditionals and Validity. Here conditions for a valid argument remain unchanged: an argument is valid if (and only if) any valuation simultaneously satisfying the premises also satisfies the conclusion. Truth tables thus establish the validity of the following argument.

1. If Rex's team lost, then Rex is upset.
2. Rex's team lost.

∴ Rex is upset.

$$(P \rightarrow Q)$$
$$P$$
$$\therefore Q$$

(2) P	Q	**(1)** $(P \rightarrow Q)$	∴ Q
1	1	**1**	**1**
1	0	0	0
0	1	1	1
0	0	1	0

But this next argument is invalid.

1. If Rex's team lost, then Rex is upset.
2. Rex is upset.

∴ Rex's team lost.

$(P \rightarrow Q)$. Q ∴ P

P	**(2)** Q	**(1)** $(P \rightarrow Q)$	∴ P
1	1	1	1
1	0	0	1
0	**1**	**1**	**0**
0	0	1	0

But with conditionals in hand we can draw a connection that would not have been so obvious previously. First, note that each argument has a corresponding sentence: a conditional with premises as antecedent, and conclusion as consequent.[1] (If the argument has more than one premise, the *conjunction* of these premises forms the antecedent.) So the above valid argument has the following "**conditional counterpart**".

$\underline{(P \rightarrow Q) \;.\; P} \;\; \therefore \; Q$

$(\underline{(P \rightarrow Q) \wedge \; P} \,) \rightarrow Q \,)$

And the invalid argument has this 'conditional counterpart'.

$\underline{(P \rightarrow Q) \;.\; Q} \;\; \therefore \; P$

$(\underline{(P \rightarrow Q) \wedge \; Q} \,) \rightarrow P \,)$

Second, consider the semantic profile of each of these conditionals. For the **valid** argument, its conditional counterpart is a **tautology**.

$$\mathbf{(P \rightarrow Q) \;.\; P \;\; \therefore \;\; Q}$$

P	Q	$(P \rightarrow Q)$	$((P \rightarrow Q) \wedge P)$	$\mathbf{((P \rightarrow Q) \wedge P) \rightarrow Q)}$
1	1	1	1	**1**
1	0	0	0	**1**
0	1	1	0	**1**
0	0	1	0	**1**

On reflection that should come as no surprise. For a valid argument is one where no valuation makes the premises true and conclusion false. But with premises serving as antecedent and conclusion as consequent, this becomes: no valuation makes antecedent true and consequent false. Since that is the only sort of valuation which makes a conditional false, our conditional is thus guaranteed to be false in no valuation – hence a tautology.

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[1] Technically: since we take each conditional (indeed, each formal sentence) to be only finitely long, only an argument with finitely many premises will have a conditional counterpart. Were we to allow an argument with *infinitely* many premises, such an argument would not have a conditional counterpart.

For the **invalid** argument, its conditional counterpart is **not a tautology**.

$$\mathbf{(P \rightarrow Q)\ .\ Q\ \therefore\ P}$$

P	Q	$(P \rightarrow Q)$	$((P \rightarrow Q) \wedge Q)$	$\mathbf{((P \rightarrow Q) \wedge Q) \rightarrow P)}$
1	1	1	1	**1**
1	0	0	0	**1**
0	1	1	1	**0**
0	0	1	0	**1**

This too stands to reason. The argument was invalid because there is at least one validity counterexample – a valuation making all the premises true, and the conclusion false. But that valuation will likewise make the antecedent of the conditional true, and its consequent false – rendering the whole conditional **false** in that valuation. And since the conditional is false in at least one valuation, it is *not* a tautology.[2]

This result holds in general.

> Each argument has a “**conditional counterpart”**: a conditional with the premise(s) of the argument (conjoined together) as antecedent, and conclusion of the argument as consequent.
>
> An argument is **valid** if (and only if) its conditional counterpart is a **tautology**.

The parallel between arguments and conditionals works both ways, of course. For each conditional likewise has an “argument counterpart” with the antecedent of the conditional as its (one) premise, and consequent as its conclusion. And the same link holds there between tautology (of the conditional) and validity (of the argument).

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[2] We formed a **conjunction** out of multiple premises precisely to guarantee this match between tautology and validity. For a valid argument, the conclusion must be true whenever **all the premises are true**; and in a validity counterexample the conclusion is false while **all the premises are true**. Since a conjunction is true only when **all its parts are true**, the conjoining of all the premises together is true when (and only when) all the premises are true.

Despite this striking parallel between arguments and conditionals, however, it would be a mistake to view conditionals as arguments, or arguments as conditionals. For I do not stake the same claim when asserting an argument as I do when asserting its conditional counterpart. Asserting an argument will, in the bargain, assert both the premise(s) and the conclusion.

1. I won the lottery.

∴ I'm a millionaire.

The only way I can sincerely assert the above argument is by asserting that *I won the lottery*, and that *I'm a millionaire*.

But when I assert the 'conditional counterpart' of this argument, I don't assert either of these claims.

If I won the lottery, then I'm a millionaire.

In saying this conditional I don't claim that *I won the lottery*, nor that *I'm a millionaire* – only that there's a link between the one event's holding and the other's.

So we continue to recognize a difference between arguments and conditionals. But we now also recognize a close link between the two.

2. Biconditionals and Logical Equivalence. Recall that when two sentences are **logically equivalent** (have the same truth table), each sentence will follow validly from the other.

For instance, "P" and "~~P" are logically equivalent; and each follows validly from the other.

P	**~P**	**~~P**
1	0	**1**
0	1	**0**

Valid

P

∴ ~~P

Valid

~~P

∴ P

Applying the above moral about conditional counterparts, that means: the conditional counterparts of each argument is a **tautology**.
Truth tables bear this out.

P	**~P**	**~~P**	**(P → ~~P)**	**(~~P → P)**
1	0	1	**1**	**1**
0	1	0	**1**	**1**

But since the second conditional is the converse of the first, the two conditionals together are equivalent to a **biconditional**. [3] And indeed, the biconditional made from “P” and “~~P” is itself a tautology.

P	**~P**	**~~P**	**(P ↔ ~~P)**
1	0	1	**1**
0	1	0	**1**

This point holds in general.

> Two sentences are **logically equivalent** if (and only if) the **biconditional** built from those two sentences is a **tautology**.

When two sentences are not logically equivalent, their corresponding biconditional is not a tautology. For instance, “(P ∧ Q)” and “P” are *not* logically equivalent; and their corresponding biconditional is *not* a tautology.

P	**Q**	**(P ∧ Q)**	**((P ∧ Q) ↔ P)**
1	1	1	1
1	0	**0**	**0**
0	1	0	1
0	0	0	1

[3] A biconditional is equivalent to the **conjunction** of the conditional and its converse; and since a conjunction is true just when both its parts are true, the biconditional is true just when both the conditional and its converse are true.

3. Tautology and Consistency (Again). The above points provide a striking consolidation of our semantic concepts. For we originally treated as three separate matters (i) whether an argument is valid; (ii) whether two sentences are logically equivalent; and (iii) whether a sentence is a tautology. But with conditionals and biconditionals in hand, we see that the first two can be swept under the carpet of the third: testing a sentence for 'tautology-hood' by itself also serves as a test of validity or of logical equivalence. Somewhat surprisingly, perhaps, *being a tautology* seems to form the core concept of logic.

It would be more accurate, however, to say that introducing conditionals has instead provided a new way of thinking about some familiar semantic observations.

(I) We noted in the previous chapter[4] that the concept of **consistency** can be used to provide a new definition for "validity".

> **Counterexample Set** (for an argument): the set
> {Premises, Negation of Conclusion}

> **Valid argument**: an argument whose counterexample set is **inconsistent**.

But "counterexample set" was later translated into its sentence counterpart, the **counterexample sentence** for an argument.[5]

> **Counterexample Sentence** (for an argument): the conjunction of all the premises, and the negation of the conclusion, of that argument.

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[4] In Section 3.17. *Validity and Inconsistency*.
[5] In Section 3.28, *Sentence Analysis*.

Once again the argument is valid if (and only if) its counterexample sentence is inconsistent (i.e., a contradiction). So the following argument is (again) valid.

$$(\mathbf{P} \rightarrow \mathbf{Q}) \;\; . \;\; \mathbf{P} \;\; \therefore \;\; \mathbf{Q}$$

P	Q	~Q	(P → Q)	((P → Q) ∧ P)	**((P → Q) ∧ P) ∧ ~Q)**
1	1	0	1	1	**0**
1	0	1	0	0	**0**
0	1	0	1	0	**0**
0	0	1	1	0	**0**

And this argument is (again) invalid, since its counterexample sentence is consistent (satisfied in Valuation 3).

$$(\mathbf{P} \rightarrow \mathbf{Q}) \;\; . \;\; \mathbf{Q} \;\; \therefore \;\; \mathbf{P}$$

P	Q	~P	(P → Q)	((P → Q) ∧ Q)	**((P → Q) ∧ Q) ∧ ~P)**
1	1	0	1	1	**0**
1	0	0	0	0	**0**
0	1	1	1	1	**1**
0	0	1	1	0	**0**

(II) Since the negation of a contradiction is itself a tautology, we can extend that last point: an **argument is valid** if (and only if) the **negation of its counterexample sentence** is a **tautology**.

But one more semantic observation brings these meditations full circle. Recall that a conditional is equivalent to the negation of a specific conjunction – namely, the conjunction of the antecedent and negation of the consequent.

"If Rex goes out, he'll take his umbrella": (P → Q)

"**It is not the case that** Rex will go out **without** taking his umbrella"
("Rex wo**n't** go out **without** taking his umbrella"): ~(P ∧ ~Q)

That means the negation of the counterexample sentence is likewise equivalent to a conditional: the conditional with the conjunction of argument's premises as antecedent, and conclusion of the argument as consequent.

But that's just the argument's conditional counterpart all over again. Just as the argument is valid if (and only if) the negation of its counterexample sentence is a tautology, so **(III)** the argument is valid if (and only if) its conditional counterpart is a tautology.

While the conditional counterpart seemed at first to offer a new take on validity, we now recognize it as just a restatement of the familiar consistency approach.

**Summary: Conditionals and Validity,
Biconditionals and Logical Equivalence**

- Each argument has a "**conditional counterpart"**: a conditional with the premise(s) of the argument (conjoined together) as antecedent, and conclusion of the argument as consequent.

- An argument is **valid** if (and only if) its conditional counterpart is a **tautology**.

- Two sentences ● and ▲ are **logically equivalent** if (and only if) the biconditional (● ↔ ▲) is a **tautology**.

- An argument's conditional counterpart is equivalent to the negation of its counterexample sentence. So the conditional counterpart is a tautology if (and only if) its counterexample sentence is a contradiction.