2.29. Conjunctive and Disjunctive Normal Forms

1. Conjunctive Normal Form. We've seen that, armed with sentences in Disjunctive Normal Form (DNF), we can find an appropriate formal sentence for any given truth table. As noted earlier, since DNF sentences embedded basics within conjunctions, and those conjunctions within disjunctions, DNF imposes a **hierarchy of scope** on our three connectives: a tilde takes only a sentence letter as its scope sentence; a conjunction takes only basics as scope; and a disjunction, with the widest scope of the three, takes those basics and conjunctions of basics as its scope.

But switching conjunctions and disjunctions in that hierarchy yields a different family of sentences, in **Conjunctive Normal Form** (**CNF**). CNF embeds basics within disjunctions, and basics or disjunction of them within conjunctions.

Basics:

1. Sentence letters are basics.
2. Negations of sentence letters are basics.

Basic Disjunctions:

1. Basics are basic disjunctions.
2. If ● and ▲ are basic disjunctions,
   then (● ∨ ▲) is a basic disjunction.

Sentences in Conjunctive Normal Form (CNF):

1. Basic Disjunctions are CNF sentences
2. If ● and ▲ are CNF sentences,
   then (● ∧ ▲) is a CNF sentence.

The basics and basic disjunctions of old are in CNF; so the following are all CNF sentences.

P	$(P \vee \sim P)$
$\sim P$	$(P \vee (\sim P \vee \sim Q))$
$(P \vee Q)$	$(\sim P \vee (\sim Q \vee R))$

Note that these also qualify as DNF sentences. Moreover, the basic conjunctions (however-many-barreled conjunctions of basics) which appeared in DNF also qualify as CNF sentences (again counting basics as mutant, one-part disjunctions). So all of these are CNF sentences.

$(P \wedge \sim P)$
$(P \wedge Q \wedge \sim R)$
$(P \wedge Q \wedge \sim R \wedge \sim S)$

CNF and DNF part company when sentences have both wedges and vels. The following sentence, for example, qualifies as CNF but not DNF.

$$(P \vee Q \vee R) \wedge (P \vee \sim Q \vee R) \wedge (\sim P \vee Q \vee \sim R)$$

Basics, basic conjunctions, and basic disjunctions thus form the common core of these two different sets of sentences.

Let us here introduce a bit of simplifying jargon: call the parts of a DNF sentence being disjoined together the **cell(s)** of that DNF sentence. So of the following DNF sentences the first sentence has three cells, the second has four cells, and the third (lacking any vels) is one-celled.

$\underline{(P \wedge \sim P)} \vee \underline{(Q \wedge \sim Q)} \vee \underline{(R \wedge \sim R)}$
$\underline{(\sim P \wedge Q)} \vee \underline{(\sim Q \wedge R)} \vee \underline{(R \wedge S)} \vee \underline{(\sim S \vee P)}$
$P \wedge Q \wedge R \wedge S$

The parts of a CNF sentence being conjoined together will likewise count as cells. So the following are (respectively) a two-celled, a three-celled, and a one-celled CNF sentence.

$\underline{(\sim P \lor Q \lor R)} \land \underline{(P \lor Q \lor \sim R)}$
$\underline{(P \lor Q)} \land \underline{(\sim P \lor Q)} \land \underline{(\sim P \lor \sim Q)}$
$P \lor \sim P \lor Q$

Returning to our earlier point about the overlap between these two families of sentences: the following counts as both a one-celled DNF sentence and a four-celled CNF sentence.

$P \land Q \land R \land S$

Whereas the next sentence is both a three-celled DNF sentence and a one-celled CNF sentence.

$P \lor \sim P \lor Q$

2. Conjunctive Normal Form and Expressive Adequacy. By constructing a procedure to match each truth table with a CNF sentences, we can demonstrate that CNF also forms an expressively adequate family of sentences.

The procedure begins just as the DNF method did: attaching to a mystery truth table with $\mathbf{2^N}$ many valuations the truth tables for **N** many sentence letters. So the following truth table, with 4 valuations, takes two sentence letters, "P" and "Q".

P	**Q**	**?**
1	1	1
1	0	0
0	1	0
0	0	1

But now instead of picking out the 'true' valuations (those with a 1), we pick out the 'false' ones (those with a 0). For each such 'false' valuation we construct a **counter-valuation sentence** – a disjunction – according to this procedure.

- If the sentence letter is true in that valuation, the counter-valuation sentence **includes the negation of that sentence letter**.
- If the sentence letter is false in that valuation, the counter-valuation sentence **includes that sentence letter**.

So for the second valuation the matching sentence is "(~P ∨ Q)," and for the third valuation "(P ∨ ~Q)".

P	**Q**	**~P**	**~Q**	**?**	**(~P ∨ Q)**	**(P ∨ ~Q)**
1	1	0	0	1	1	1
1	**0**	0	1	**0**	**0**	1
0	**1**	1	0	**0**	1	**0**
0	0	1	1	1	1	1

Conjoining these counter-valuation sentences yields a two-celled CNF sentence taking our mystery truth table.

P	**Q**	**~P**	**~Q**	**?**	**(~P ∨ Q)**	**(P ∨ ~Q)**	**(~P ∨ Q) ∧ (P ∨ ~Q)**
1	1	0	0	1	1	1	1
1	0	0	1	**0**	**0**	1	**0**
0	1	1	0	**0**	1	**0**	**0**
0	0	1	1	1	1	1	1

Generally each sentence letter at the left of the table appears exactly once in each counter-valuation sentence – as in the above example. So counter-valuation sentences and conjunctions of them will cover any mystery truth table except one: where the truth table has 1 in **every** valuation. In that case any tautology – such as "(P ∨ ~P)" – fills the bill. (Since "(P ∨ ~P)" counts as a one-celled CNF sentence, choice of this sentence does not take us outside CNF.)

Thus a general procedure for matching a CNF sentence to a truth table works like so.

- If the truth table is false in exactly one valuation, build a counter-valuation sentence false in that valuation.
- If the truth table is false in more than one valuation, build a counter-valuation conjunction (a conjunction of counter-valuation sentences) false in those valuations.
- If the truth table is false in no valuation, use “(P ∨ ~P)” as the matching sentence.

As this covers every possible sort of truth table, CNF provides an **expressively adequate** family of sentences.

3. CNF Meets DNF. Since DNF is likewise an expressively adequate family of sentences, each truth table will be matched with both a CNF and a DNF sentence. That means each DNF sentence has a corresponding CNF sentence. And in fact we can, provided a CNF or DNF sentence, construct its counterpart without appeal to truth tables. Such a procedure relies solely on the following formal equivalence, noted in our earlier discussion of the 3D Method.[1]

Distribution: “(● ∧ (▲ ∨ ◆))” ≡ “((● ∧ ▲) ∨ (● ∧ ◆))”
“(● ∨ (▲ ∧ ◆))” ≡ “((● ∨ ▲) ∧ (● ∨ ◆))”

For instance, we can begin with the two-celled CNF sentence constructed above.

1. $\mathbf{(\sim P \lor Q) \land (P \lor \sim Q)}$

Distribution pushes the left part of this conjunction, along with the wedge, into the right disjunction.

1. $\mathbf{\underline{(\sim P \lor Q) \land}\,(P \lor \sim Q)}$
2. $\mathbf{(\underline{(\sim P \lor Q) \land}\,P) \lor (\underline{(\sim P \lor Q) \land}\,\sim Q)}$

[1] In 2.28.

A second round of distribution pushes the right part of each conjunction into its neighboring disjunction.

1. $\mathbf{(\sim P \lor Q) \land (P \lor \sim Q)}$
2. $\mathbf{(\underline{(\sim P \lor Q) \land}\ P) \lor (\underline{(\sim P \lor Q) \land}\ \sim Q)}$
3. $\mathbf{((\sim P \underline{\land P}) \lor (Q \underline{\land P})) \lor ((\sim P \underline{\land \sim Q}) \lor (Q \underline{\land \sim Q}))}$

Sentence (3) is a four-celled DNF sentence. Extraneous parentheses can be left off.

3. $\mathbf{(\sim P \land P) \lor (Q \land P) \lor (\sim P \land \sim Q) \lor (Q \land \sim Q)}$

This DNF sentence does indeed take the same truth table as the original in CNF. (Since a disjunction is true wherever at least one of its parts is true, the disjunction is true in valuations 1 and 4.)

P	**Q**	**~P**	**~Q**	**(~P ∨ Q)**	**(P ∨ ~Q)**	**(~P ∨ Q) ∧ (P ∨ ~Q)**
1	1	0	0	1	1	**1**
1	0	0	1	0	1	**0**
0	1	1	0	1	0	**0**
0	0	1	1	1	1	**1**

(~P ∧ P)	**(P ∧ Q)**	**(~P ∧ ~Q)**	**(Q ∧ ~Q)**	**(~P ∧ P) ∨ (P ∧ Q) ∨ (~P ∧ ~Q) ∨ (Q ∧ ~Q)**
0	**1**	0	0	**1**
0	0	0	0	**0**
0	0	0	0	**0**
0	0	**1**	0	**1**

And since the Distribution Law is an equivalence, holding in both directions, we could just as easily started with a DNF sentence and convert it into CNF.

1. $\mathbf{(Q \land P) \lor (\sim P \land \sim Q)}$

Two applications of Distribution yield a four-celled sentence in CNF.

2. $\mathbf{((Q \land P) \lor \sim P) \land ((Q \land P) \lor \sim Q)}$
3. $\mathbf{(Q \lor \sim P) \land (P \lor \sim P) \land (Q \lor \sim Q) \land (P \lor \sim Q)}$

4. DNF and CNF: Simplification and Evaluation. We can simply some of those last results – as well as better appreciate features of DNF and CNF sentences – by noting points about the sentences serving as their cells.

First, DNF sentences have as cells basics, and basic conjunctions built out of these. Now basic conjunctions follow the general semantic rule governing all conjunctions: **a conjunction is true only where all its parts are true**. Since no valuation makes both a sentence and its negation true, a basic conjunction containing a sentence letter and its negation is a contradiction.

> **A basic conjunction containing a sentence letter and the negation of that sentence letter is a contradiction**.

Thus we can tell that the following basic conjunctions are contradictions without appeal to truth tables.

$P \wedge \sim Q \wedge R \wedge \sim P$
$R \wedge S \wedge \sim R$

But note further a general point about disjunctions: **disjoining together some sentence S and a contradiction is equivalent to sentence S alone**.

So since “$(Q \wedge \sim Q)$” is a contradiction, “$P \vee (Q \wedge \sim Q)$” is equivalent to “P”.

P	**Q**	**~Q**	**$(Q \wedge \sim Q)$**	**$P \vee (Q \wedge \sim Q)$**
1	1	0	0	**1**
1	0	1	0	**1**
0	1	0	0	**0**
0	0	1	0	**0**

That allows us to simplify DNF sentences: any contradictory cell – basic conjunction containing a sentence letter and its negation – can be eliminated without semantic change to that DNF sentence.

So earlier, beginning with the two-celled CNF sentence "(~P ∨ Q) ∧ (P ∨ ~Q)," we obtained the following four-celled DNF sentence via Distribution.

<u>(~P ∧ P)</u> ∨ (Q ∧ P) ∨ (~P ∧ ~Q) ∨ <u>(Q ∧ ~Q)</u>

But since the first and fourth cells here are contradictions (containing a sentence letter and its negation), the sentence as a whole is logically equivalent to this simpler DNF sentence.

(Q ∧ P) ∨ (~P ∧ ~Q)

And that simpler sentence is indeed equivalent to the original CNF sentence.

P	**Q**	**~P**	**~Q**	**(~P ∨ Q)**	**(P ∨ ~Q)**	**(~P ∨ Q) ∧ (P ∨ ~Q)**
1	1	0	0	1	1	**1**
1	0	0	1	0	1	**0**
0	1	1	0	1	0	**0**
0	0	1	1	1	1	**1**

(Q ∧ P)	**(~P ∧ ~Q)**	**(Q ∧ P) ∨ (~P ∧ ~Q)**
1	0	**1**
0	0	**0**
0	0	**0**
0	1	**1**

A further consequence: if **every** cell of a DNF sentence is a contradiction, then the whole DNF sentence is a contradiction. For a disjunction is false only when all its parts are false; but with a disjunction all of whose cells are contradictions, all those parts will be false in every valuation.

If every cell of a DNF sentence is a contradiction, the whole DNF sentence is a contradiction.

And again, we can tell that a cell is a contradiction by its having a sentence letter and its negation.

So we know that the following DNF sentences are contradictions withoutresorting to truth tables.

$(P \wedge \sim P \wedge Q) \vee (\sim Q \wedge R \wedge Q) \vee (P \wedge R \wedge \sim P)$
$(P \wedge X \wedge \sim P) \vee (\sim S \wedge S)$
$(P \wedge \sim P \wedge T)$

Parallel morals hold for CNF sentences. First, CNF sentences have as cells basics and basic disjunctions. And like all disjunctions, a basic disjunctions, **a basic disjunction is false only when all its parts are false**. But since no valuation makes both a sentence and its negation false, a basic disjunction containing a sentence letter and its negation will be a tautology.

> **A basic disjunction containing a sentence letter and its negation is a tautology**.

So we can tell that the following basic disjunctions are tautologies without appeal to truth tables.

$P \vee \sim Q \vee R \vee \sim P$
$R \vee S \vee \sim R$

Note further that **the conjunction of some sentence with a tautology is logically equivalent to that sentence alone**. So “$(P \wedge (Q \vee \sim Q))$” is equivalent to “P”.

P	**Q**	**~Q**	**(Q ∧ ~Q)**	**(P ∨ (Q ∧ ~Q))**
1	1	0	0	**1**
1	0	1	0	**1**
0	1	0	0	**0**
0	0	1	0	**0**

So any tautological cell of a CNF sentence – a cell containing a sentence letter and its negation –can be removed without semantic change to that CNF sentence. For instance, the first and third cells of the following conjunction are tautologies.

$\mathbf{\underline{(\sim P \vee P)} \wedge (Q \vee P) \wedge \underline{(Q \vee \sim Q)} \wedge (\sim P \vee \sim Q)}$

The CNF sentence is thus semantically equivalent to this simpler conjunction.

<u>(~P ∨ P)</u> ∧ (Q ∨ P) ∧ <u>(Q ∨ ~Q)</u> ∧ (~P ∨ ~Q)

(Q ∨ P) ∧ (~P ∨ ~Q)

As an extension of that point: if every cell of CNF sentence is a tautology – containing both a sentence letter and its disjunction – then the whole CNF sentence is a tautology. For a conjunction is only truth when all its parts are true; but with every cell a tautology, every valuation makes the whole conjunction true.

> **If every cell of a CNF sentence is a tautology, then the whole CNF sentence is a tautology**.

That means, for instance, that all of the following are tautologies.

(P ∨ ~P ∨ Q) ∧ (~Q ∨ R ∨ Q) ∧ (P ∨ R ∨ ~P)
(P ∨ X ∨ ~P) ∧ (~S ∨ S)
(P ∨ ~P ∨ T)

While the previous observations have been about when a DNF sentence is a contradiction and when a CNF sentences is a tautology, we close with a brief point about the opposite sort of cases: where a DNF sentence is a tautology, or a CNF sentence a contradiction.

Note first that if a DNF sentence contains no wedges, it is one-celled. In that case we can apply the point about disjunctions just rehearsed: since a disjunction is true so long as one of its parts is true, such a one-celled DNF sentence is a tautology if (and only if) it contains a sentence letter and the negation of that sentence letter.

> **A DNF sentence with no wedges is a tautology if (and only if) it contains a sentence letter and the negation of that sentence letter.**

Likewise a CNF sentence with no vels is one-celled, in which case the earlier semantic point applies: since a conjunction is false whenever at least one part is false, such a one-celled CNF sentence is a contradiction if (and only if) it contains a sentence letter and its negation.

> **A CNF sentence with no vels is a contradiction if (and only if) it contains a sentence letter and the negation of that sentence letter.**

Finally, thanks to the 3D Method and the above methods for simplifying DNF and CNF, we combine our observations into a **general method** for deciding – for any Chapter Three sentence – whether it’s a tautology, a contradiction, or neither.

- If the sentence converts into a DNF sentence whose every cell contains some sentence letter and its negation, then the sentence is a contradiction.
- If the sentence converts into a CNF sentence whose every cell contains some sentence letter and its negation, then the sentence is a tautology.
- If neither result occurs, then the sentence is neither a contradiction nor a tautology.

And once again, this general test proceeds entirely without appeal to truth tables or truth trees.

Summary: Conjunctive and Disjunctive Normal Forms

- A sentence in **Conjunctive Normal Form (CNF)** is a (however-many-place) conjunction of (however-many-place) disjunctions of basics**.**

- CNF forms an **expressively adequate** family of sentences.

- For each CNF sentence there is a logically equivalent sentence in **Disjunctive Normal Form (DNF)**.

- A sentence in DNF is a **contradiction** if (and only if) each of its cells contains a sentence letter and also the negation of that sentence letter.

- A sentence in CNF is a **tautology** if (and only if) each of its cells contains a sentence letter and also the negation of that sentence letter.