2.34. More Duals

> This symmetry, or *duality* as it is called, which exists in principles and definitions, must also exist in all the formulas deduced from them as long as no principle or definition is introduced which would overthrow them. Hence a true formula may be deduced from another true formula by transforming it by the principle of duality…. In its application the *law of duality* makes it possible to replace two demonstrations by one.
>
> – Louis Couturat, **The Algebra of Logic** (1905)

To appreciate the importance of duality to the concerns of logic, we here survey some of the ways duality is woven through key concepts familiar from previous sections. For this makes especially clear the ‘shortcuts’ duality offers: if features and families of sentences are guaranteed a logical ‘mirror image’, testing for such a feature, or membership in a certain family, will automatically bring with it a test for the dual of that feature or family.

1. Duals of Sentences and Arguments. It will hold in general that **the connective dual of a tautology is a contradiction**. That is clear already from the True/False Swap (which always accompanies the Connective Swap): the truth table for a tautology (true in every valuation) becomes the truth table for a contradiction (false in every valuation), and vice versa. For instance, the connective dual of the tautology “(P ∨ ~P)” is the contradiction “(P ∧ ~P)”.

True	**False**
Disjunction	**Conjunction**
Tautology	**Contradiction**

Negation

(And any sentence which is **neither a tautology nor** a **contradiction** will have a dual sentence likewise neither tautology nor contradiction.)

Next, note that **a sentence in Disjunctive Normal Form has a sentence in Conjunctive Normal Form as its connective dual**. (And any sentence which is in **neither DNF nor CNF** will have a dual sentence likewise neither DNF nor CNF.) So whereas, e.g., “(~P ∧ Q) ∨ (P ∧ ~Q)” is in DNF, its dual “(~P ∨ Q) ∧ (P ∨ ~Q)” is in CNF.

True	**False**
Disjunction	**Conjunction**
Tautology	**Contradiction**
DNF Sentence	**CNF Sentence**

Negation

This observation confirms the previous point about duality. For we saw that a DNF sentence, each of whose “cells” (basic conjunctions) contains a sentence letter and also its negation, is a contradiction.[1] “(~P ∧ P) ∨ (Q ∧ ~Q)”, for instance, is a DNF contradiction. And its connective dual is the CNF sentence “(~P ∨ P) ∧ (Q ∨ ~Q)” – which, having a sentence letter and its negation in each of its “cells” (basic disjunctions), is a tautology. Once again the dual of a contradiction is a tautology.

Concerning **validity**, we note first: it is **not** in general true that if Sentence 2 follows validly from Sentence 1, then the connective dual of Sentence 2 follows validly from the dual of Sentence 1. For example, “~P” follows validly from “~(P ∨ Q)”; but the connective dual of “~P” (just “~P” again) doesn’t follow validly from the connective dual of “~(P ∨ Q)” – namely “~(P ∧ Q)”.

VALID	**INVALID**
~(P ∨ Q)	~(P ∧ Q)
∴ ~P	∴ ~P

[1] In 2.29.

But if **premise and conclusion change places** in the ‘dual argument’ then validity is preserved under (connective and semantic) duality: if Sentence 2 follows validly from Sentence 1, the dual of Sentence 1 follows from the dual of Sentence 2. For instance, “~(P ∧ Q)” follows validly from “~P”.

VALID	**VALID**
~(P ∨ Q)	~P
∴ ~P	∴ ~(P ∧ Q)

And we can see that holds in general, if we keep in mind that True is the dual of False. For to say that Sentence 2 follows validly from Sentence 1 – that is, that the argument **Sentence 1 ∴ Sentence 2** is valid – is just to say that there is no validity counterexample, hence no valuation where Sentence 1 is true but Sentence 2 is false. But under the True/False Swap that claim becomes: there is no valuation where Dual of Sentence 1 is false but Dual of Sentence 2 is true.[2] That means there is no validity counterexample for the argument **Dual of Sentence 2 ∴ Dual of Sentence 1**.[3]

> **If Sentence 2 follows validly from Sentence 1, then the (connective or semantic) dual of Sentence 1 follows validly from the dual of Sentence 2.**

VALID	**VALID**
Sentence 1	Dual of Sentence 2
∴ Sentence 2	∴ Dual of Sentence 1

[2] Since the argument here is semantic (in terms of which situations will make a sentence true or false), the point is most obvious concerning **semantic** duals of premises and conclusions. But since the points about truth and validity hold for any semantic dual of premise or conclusion – they all, by definition, being logically equivalent – the point holds, in particular, for the connective dual of those premises and conclusions (since connective duality brings semantic duality in its wake).

[3] Borrowing an argument from (Quine 1959: 62).

That provides, in particular, a definition of **connective dual of an argument**.

> The connective **dual of an argument** is the result of (i) switching the conclusion and the premise(s) of that argument, then (ii) replacing each sentence with its connective dual. (If the argument has more than one premise, these premises are conjoined together before Step (i).)

So the dual of argument “$\sim(P \vee Q)\ \therefore\ \sim P$” is “$\sim P\ \therefore\ \sim(P \wedge Q)$”; and the dual of the argument “$(P \vee Q)\ .\ \sim P\ \therefore\ Q$” is “$Q\ \therefore\ ((P \wedge Q) \vee \sim P)$”.[4]

From the previous point about the validity of dual arguments, it’s easy to see that **logical equivalence is preserved under (semantic or connective) duality**. For two sentences are logically equivalent if (and only if) each follows validly from the other.[5]

> **If two sentences are logically equivalent, their duals are logically equivalent.**

So, for example, from either half of DeMorgan’s Law we can derive the other half by duality.

“$\sim(P \wedge Q)$” **is equivalent to** “$(\sim P \vee \sim Q)$”
“$\sim(P \vee Q)$” **is equivalent to** “$(\sim P \wedge \sim Q)$”

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[4] In the second example, the premises “$(P \vee Q)\ .\ \sim P$” are first conjoined together, yielding “$((P \vee Q) \wedge \sim P)$”; then this conjunction (put in the conclusion spot) is replaced with its connective dual “$((P \wedge Q) \vee \sim P)$”.

[5] As noted in 2.20 and 2.25.

2. Duals of Valuations and Counterexamples. The points about argument validity carry over to argument invalidity. Most obviously: if an argument is invalid, then its connective dual argument is invalid as well.[6] For example, since the argument P ∴ (P ∧ Q) is invalid, the argument (P ∨ Q) ∴ P is invalid as well.

INVALID	**INVALID**
P	(P ∨ Q)
∴ (P ∧ Q)	∴ P

But there's more to be said about an invalid argument and its dual argument. To begin, note that while we've applied the True/False Swap to whole truth tables, we can use it equally on single valuations. Just as with an entire truth table, the semantic dual of a valuation is just the True/False Swap of that valuation.

Semantic Dual of a Valuation: the True/False Swap of that valuation

For instance, a valuation with "P" and "R" true and "Q" false will have as its semantic dual a valuation with "P" and "R" false and "Q" true.

Valuation:

P	**Q**	**R**
1	0	1

Dual of Valuation:

P	**Q**	**R**
0	1	0

Now an invalid argument has at least one **validity counterexample** – a valuation where the premises of the argument are true but the conclusion is false. And if Argument A has valuation V_A as a counterexample, the dual argument D(A) has as counterexample the dual of valuation V_A, $D(V_A)$.

[6] If that weren't the case – if there were some **invalid argument, A**, whose dual argument D(A) was nonetheless valid – then from the previous section we know the dual of D(A) would also be valid. But (thanks to the involutary nature of connective duality) the dual of D(A) is just argument A again. So A would be valid, contradicting the original claim that A is invalid. Moral: there's no consistent way for an argument to be invalid while its dual argument is valid.

If an argument has a validity counterexample, then the connective dual of that argument has the semantic dual of that valuation as a validity counterexample.

This is clear from the True/False Swap: the valuation V_A that's a validity counterexample for Argument A makes the (conjoined) premise of the argument true and the conclusion false, and so (under the True/False Swap) makes the dual of that premise false and the dual of that conclusion true. But since premise and conclusion switch places in the dual of an argument, this dual valuation makes the premise of the dual argument true and the conclusion false – hence serving as a counterexample for the dual argument.

For example, the counterexample for the argument P $\therefore$ (P $\wedge$ Q) is the second valuation, where **"P" is true and "Q" is false**. And the counterexample of the dual argument (P $\vee$ Q) $\therefore$ P is the semantic dual of that valuation, where **"P" is false and "Q" is true**.

P	Q	$\therefore$ **(P $\wedge$ Q)**
1	1	1
1	0	**0**
0	1	0
0	0	0

P	Q	**(P $\vee$ Q)**	$\therefore$ **P**
0	0	1	0
0	1	**1**	**0**
1	0	1	1
1	1	0	1

3. Duality and Negation. It is notable that many of the features of duality listed are features shared with the semantics of **negation**. For the True/False Swap enacts the same semantic behavior as negation: replacing True with False and False with True. From that it is immediately obvious that **negation**, like the True/False Swap, is semantically **involutary**: the negation of the negation of a sentence is semantically equivalent to the original sentence.

The earlier observation about valid arguments and duality likewise holds for negations.

> **If Sentence 2 follows validly from Sentence 1, then the negation of Sentence 1 follows validly from the negation of Sentence 2.**

For example: “~P” follows validly from “~(P ∨ Q)”; and the negation of “~(P ∨ Q)” – equivalently, “(P ∨ Q)” – follows validly from the negation of “~P”, semantically equivalently to “P”.

And if two sentences are logically equivalent, their respective negations are also logically equivalent. Likewise, just as the dual of a tautology is a contradiction (and vice versa), so the negation of a tautology is a contradiction (and vice versa).

Indeed, we could construct dual sentences by way of negation rather than through Connective Swap. For when the True/False Swap changes true to false and false to true in truth tables – for the sentence letters, and for the final, completed sentence – it is effecting the same change which negation does. That means we could achieve the same semantic result in truth tables if we added tildes – again, to each sentence letter, and to the whole sentence.

For instance, we know already that the True/False Swap transforms the truth table for “(P ∨ Q)” into the truth table for “(P ∧ Q)”. But if we add a tilde to each sentence letter in “(P ∨ Q)” and a tilde to the whole (final) sentence –yielding “~(~P ∨ ~Q)” – we have a sentence which takes the same dual truth table picked out by the True/False Swap.

P	Q	**(P ∨ Q)**	~P	~Q	(~P ∨ ~Q)	**~(~P ∨ ~Q)**	**(P ∧ Q)**
1	1	**1**	0	0	0	**1**	**1**
1	0	**1**	0	1	1	**0**	**0**
0	1	**1**	1	0	1	**0**	**0**
0	0	**0**	1	1	1	**0**	**0**

Call this the “Tilde Insertion” method for constructing the dual of a sentence.

> The **Tilde Insertion Dual** of a sentence is the result of placing a tilde before each sentence letter in the sentence, and before the entire sentence.

But while Tilde Insertion reasonably qualifies as the dual of a sentence – in that there is just one Tilde Insertion dual for each sentence, and it always takes the correct dual truth table (the True/False Swap of the original sentence’s truth table) – Tilde Insertion has less to recommend it than Connective Swap. For Tilde Insertion duality, unlike Connective Swap, is not involutary. For instance: the Tilde Insertion dual of “(P ∨ Q)” is “~(~P ∨ ~Q)”. But the Tilde Insertion dual of that sentence isn’t “(P ∨ Q)” again, but “~~ (~~P ∨ ~~Q)”. With Tilde Insertion, the dual of the dual isn’t the original sentence.[7]

That said, Tilde Insertion has some useful applications when exploring sentence duality in particular languages. It will, for one, prove useful for calculating the semantic dual of a sentence in cases (such as in the languages of later chapters) where truth tables don’t suffice for sentence semantics.

[7] If we added to the Tilde Insertion approach that we **remove double negations** and **apply (Inward) DeMorgan’s Law** whenever possible, the Tilde Insertion approach would usually yield the same results as the Connective Swap, and would be (to that degree) involutary. Call this the “**Simplified Tilde Insertion**” dual of a sentence. Still, certain cases won’t be involutary. For instance, the Simplified Tilde Insertion dual of “~~P” is “P” (since adding a tilde before the sentence letter in “~~P” and before the whole sentence yields “~~~~P,” which when cleared of double negations is “P”). But the Simplified Tilde Insertion dual of “P” is “P” again, not “~~P”.

As a second application, note that so long as a formal language contains the tilde (or some semantic equivalent), we can build a dual of any sentence in that language whether or not each connective finds its dual in that language. For example, the formal language {∧, ~} contains many sentences that don't find their connective dual in that language. "(P ∧ Q)," for instance, takes as connective dual the sentence "(P ∨ Q)"; but "(P ∨ Q)" isn't a sentence in the language {∧, ~}. But since {∧, ~} contains the tilde, we can construct the Tilde Insertion dual of "(P ∧ Q)" – namely, "~(~P ∧ ~Q)".

Summary: Features of Duality

More Duals:

- The (connective or semantic) **dual of tautology** is a contradiction (and vice versa).
- The connective **dual of a DNF sentence** is a CNF sentence (and vice versa)
- If Sentence S1 **validly entails** Sentence S2, then the (semantic or connective) dual of S2 validly entails the dual of S1.
- The connective **dual of an argument** is the result of (i) swapping premise(s) and conclusion (conjoining premises if there are more than one), and (ii) replacing each sentence by its connective dual.
- If two sentences are **logically equivalent**, their duals sentences are also logically equivalent.

Tilde Insertion Duality:

- The **Tilde Insertion Dual** of a sentence is the result of putting a tilde before each sentence letter in in the sentence, and before the whole sentence. The Tilde Insertion Dual of a sentence takes as its truth table the semantic dual of that sentence's truth table.