3.15. Conditionals and Biconditional: Rules and Deductions

1. Conditional Rules. In the previous chapter we coupled a general deductive strategy (using ID in general, except for cases simple enough not to need it) with an array of inference rules whose use followed a usefulness ranking. Specifically: we reach first for Elim Rules (and of them, $\wedge$– first, $\vee$– and ~– afterward), applying Intro Rules only to free up deductive logjams.

With the advent of conditionals we revise each part of that deductive strategy. As noted in the previous section, our default choice is now to use CD to deduce a conditional conclusion, and ID for all other types of conclusion – unless, in each case, we see a simple way of reaching that conclusion through inference rules alone.

Concerning inference rules, we note first that we will have no need of any Intro rule for conditionals; for CD serves the role of building up a desired conditional. But to capture all the valid arguments in the Chapter Three language we must add two **Elim rules** – bowing to tradition in using traditional Latin labels (rather than the more generic "$\rightarrow$–").[1]

Modus Ponens (MP) **Modus Tollens (MT)**

$$\begin{array}{c} (\bullet \rightarrow \blacktriangle) \\ \bullet \\ \hline \blacktriangle \end{array} \qquad\qquad \begin{array}{c} (\bullet \rightarrow \blacktriangle) \\ \sim\blacktriangle \\ \hline \sim\bullet \end{array}$$

[1] Following more recent logical tradition, we use abbreviated forms of the Latin names for these rules: "Modus Ponens" (rather than the original "Modus Ponendo Ponens") and "Modus Tollens" (rather than "Modus Tollendo Tollens").

English examples illustrate the intuitive appeal of both rules.

1. If Rex is home, the light is on.
2. Rex is home.

∴ The light is on.

1. If Rex is home, the light is on.
2. (But) The light isn't on.

∴ Rex (must) not be home.[2]

As with earlier Elim rules, **MP and MT** are to be **used whenever possible**. The following example illustrates how MP and MT (plus an earlier Elim rule) can back us into the desired sentence.

1. If we have peanut butter, then if we have jelly we can make a sandwich.
2. We have peanut butter, but we can't make a sandwich.

∴ We don't have jelly.

1. $(P \rightarrow (Q \rightarrow R))$
2. $(P \wedge {\sim}R)$

		Get: $\sim Q$
3.	P	2, $\wedge-$
4.	$\sim R$	2, $\wedge-$
5.	$(Q \rightarrow R)$	1, 3, MP
6.	$\sim Q$	4, 5, MT

2. Biconditional Rules. We saw earlier that a biconditional sentence is logically equivalent to the conjunction of a conditional and its converse – e.g., that "$(P \leftrightarrow Q)$" is equivalent to "$((P \rightarrow Q) \wedge (Q \rightarrow P))$".[3] Our Intro and Elim rules for biconditionals will reflect this, being really just Conjunction Intro and Elim in disguise.

[2] Conclusion-marking "must" is added here simply to make the example flow more naturally in English.
[3] In 3.4.

Biconditional Introduction (↔+)

$$\begin{array}{c} (\bullet \rightarrow \blacktriangle) \\ (\blacktriangle \rightarrow \bullet) \\ \hline (\bullet \leftrightarrow \blacktriangle) \end{array} \qquad\qquad \begin{array}{c} (\blacktriangle \rightarrow \bullet) \\ (\bullet \rightarrow \blacktriangle) \\ \hline (\bullet \leftrightarrow \blacktriangle) \end{array}$$

Biconditional Elimination (↔−)

$$\begin{array}{c} (\bullet \leftrightarrow \blacktriangle) \\ \hline (\bullet \rightarrow \blacktriangle) \end{array} \qquad\qquad \begin{array}{c} (\bullet \leftrightarrow \blacktriangle) \\ \hline (\blacktriangle \rightarrow \bullet) \end{array}$$

3. Deduction Strategy: Conditionals and Biconditionals. Our default strategy for deducing a conditional is to use Conditional Deduction. But we note here some further, less general tactics for deducing a conditional.

Recall that the conditional "(P → Q)" is semantically equivalent to the disjunction "(~P ∨ Q)".

P	Q	~P	**(~P ∨ Q)**	**(P → Q)**
1	1	0	**1**	**1**
1	0	0	**0**	**0**
0	1	1	**1**	**1**
0	0	1	**1**	**1**

We should therefore expect "(P → Q)" to follow from the same sentence(s) that "(~P ∨ Q)" does. But "(~P ∨ Q)" follows from "~P", and likewise from "Q", by simple ∨+.

$$\frac{\sim P}{(\sim P \lor Q)} \qquad\qquad \frac{Q}{(\sim P \lor Q)}$$

And "(P → Q)" is likewise deducible from each of these premises.

1.	~P	
		Get: (P → Q) (CD)
2.	P	ACD
3.	(P ∨ Q)	2, ∨+
4.	Q	1, 4, ∨–
5.	(P → Q)	(2, 4, CD)

1.	Q	
		Get: (P → Q) (CD)
2.	P	ACD
3.	Q	1, R
4.	(P → Q)	(2, 3, CD)

So when a certain conditional is needed in a deduction, it is tactically handy to keep in mind that it is deducible from either its consequent or the negation of its antecedent.

> ***Deduction Strategy***: A conditional is deducible from its **consequent**, as well as from the **negation of its antecedent**.

Since a biconditional is equivalent to a conjunction of two conditionals (one the converse of the other), when a biconditional is needed it can reached by first deducing the two conditionals, and then inferring the biconditional through ↔+.

Biconditional Introduction (↔+)

$$\frac{\begin{array}{l}(\bullet \rightarrow \blacktriangle)\\(\blacktriangle \rightarrow \bullet)\end{array}}{(\bullet \leftrightarrow \blacktriangle)} \qquad \frac{\begin{array}{l}(\blacktriangle \rightarrow \bullet)\\(\bullet \rightarrow \blacktriangle)\end{array}}{(\bullet \leftrightarrow \blacktriangle)}$$

For example, to deduce "(P ↔ Q)" from some premises, first deduce "(P → Q)" and "(Q → P)" using CD, and then derive the biconditional from them by **(↔+).**

> ***Deduction Strategy***: To deduce a biconditional, deduce the corresponding **conditional and its converse**, and then use **Bicon Intro (↔+)**.

Likewise, whenever we have a biconditional we know that the corresponding conditional and converse are deducible from ↔**–**.(just as, whenever we have a conjunction, we know its left and right parts are deducible by ∧**–**.) So, as with all Elim rule, we use ↔**–** whenever possible, with no forethought as to the usefulness of the conditionals this yields us.

> ***Deduction Strategy***: Use the Elim Rules MP, MT, and Bicon Elim **(**↔**–**) whenever possible.

4. Deductive Strategy: Indirect Deductions Revisited. The arrival of conditionals also brings a new wrinkle to our indirect deduction strategy. A time-saving measure for IDs was to use a sentence we already have as half of the needed contradiction, and derive the other sentence – effectively cutting our deductive work in half.

But if one of the sentences we already have is the **negation of a conditional**, the savings in labor is doubled. For then it remains only to secure the other half of the contradiction – a conditional. Since conditional deduction brings the antecedent of that conditional (as the ACD), to complete the contradiction all that remains is to deduce the other half of the conditional – its consequent.

> ***Deduction Strategy***: In an Indirect Deduction, if the **negation of a conditional** is available, **use** that **as half of the contradiction**, and deduce the other half (the conditional) using CD.[4]

As an illustration of this strategy, note that the negation of a conditional is logically equivalent to a **conjunction**: of the **antecedent**, and the **negation of the consequent**. For example, "~(P → Q)" is equivalent to "(P ∧ ~Q)".

P	Q	~Q	(P → Q)	**~(P → Q)**	**(P ∧ ~Q)**
1	1	0	1	**0**	**0**
1	0	1	0	**1**	**1**
0	1	0	1	**0**	**0**
0	0	1	1	**0**	**0**

But from "(P ∧ ~Q)" both its left and right parts, "P" and "~Q," follow immediately by ∧–. So from the equivalent "~(P → Q)," "P" and "~Q" likewise follow – as deductions show.

[4] Following a suggestion from Kalish and Montague 1964: 26 (#5).

1.	$\sim(P \rightarrow Q)$	
		~~Get:~~ P (ID)
2.	$\sim P$	AID
		~~Get~~ $(P \rightarrow Q)$ (CD)
3.	P	ACD
4.	$(P \vee Q)$	3, $\vee$**+**
5.	Q	4, 6, $\vee$**–**
6.	$(P \rightarrow Q)$	3, 5, CD
7.	$\sim(P \rightarrow Q)$	1, R
8.	P	2, 6, 7, ID

1.	$\sim(P \rightarrow Q)$	
		~~Get:~~ ~Q (ID)
2.	$\sim\sim Q$	AID
3.	Q	
		~~Get~~ $(P \rightarrow Q)$ (CD)
3.	P	ACD
4.	Q	3, R
5.	$(P \rightarrow Q)$	3, 4, CD
6.	$\sim(P \rightarrow Q)$	1, R
7.	$\sim Q$	2, 5, 6, ID

In general: whenever we have the **negation of a conditional** it is tactically useful to keep in mind that both the **antecedent** and **negation of the consequent** are each deducible from that sentence.

Inference Rules (Chapter Three)

Disjunction Rules

Disjunction Elimination ($\lor -$)

$$\begin{array}{c} (\bullet \lor \blacktriangle) \\ \sim\bullet \\ \hline \blacktriangle \end{array} \qquad\qquad \begin{array}{c} (\bullet \lor \blacktriangle) \\ \sim\blacktriangle \\ \hline \bullet \end{array}$$

Disjunction Introduction ($\lor +$)

$$\begin{array}{c} \bullet \\ \hline (\bullet \lor \blacktriangle) \end{array} \qquad\qquad \begin{array}{c} \blacktriangle \\ \hline (\bullet \lor \blacktriangle) \end{array}$$

Conjunction Rules

Conjunction Elimination ($\land -$)

$$\begin{array}{c} (\bullet \land \blacktriangle) \\ \hline \bullet \end{array} \qquad\qquad \begin{array}{c} (\bullet \land \blacktriangle) \\ \hline \blacktriangle \end{array}$$

Conjunction Introduction ($\land +$)

$$\begin{array}{c} \bullet \\ \blacktriangle \\ \hline (\bullet \land \blacktriangle) \end{array} \qquad\qquad \begin{array}{c} \blacktriangle \\ \bullet \\ \hline (\bullet \land \blacktriangle) \end{array}$$

Negation Rules

Negation Elimination ($\sim -$)

$$\begin{array}{c} \sim\sim\bullet \\ \hline \bullet \end{array}$$

Negation Introduction ($\sim +$)

$$\begin{array}{c} \bullet \\ \hline \sim\sim\bullet \end{array}$$

Repetition (R)

$$\begin{array}{c} \bullet \\ \hline \bullet \end{array}$$

Conditional Rules

Modus Ponens (MP)

$$\begin{array}{c} (\bullet \rightarrow \blacktriangle) \\ \bullet \\ \hline \blacktriangle \end{array}$$

Modus Tollens (MT)

$$\begin{array}{c} (\bullet \rightarrow \blacktriangle) \\ \sim\blacktriangle \\ \hline \sim\bullet \end{array}$$

Biconditional Rules

Biconditional Elimination (↔−)

$$\begin{array}{c} (\bullet \leftrightarrow \blacktriangle) \\ \hline (\bullet \rightarrow \blacktriangle) \end{array} \qquad \begin{array}{c} (\bullet \leftrightarrow \blacktriangle) \\ \hline (\blacktriangle \rightarrow \bullet) \end{array}$$

Biconditional Introduction (↔+)

$$\begin{array}{c} (\bullet \rightarrow \blacktriangle) \\ (\blacktriangle \rightarrow \bullet) \\ \hline (\bullet \leftrightarrow \blacktriangle) \end{array} \qquad \begin{array}{c} (\blacktriangle \rightarrow \bullet) \\ (\bullet \rightarrow \blacktriangle) \\ \hline (\bullet \leftrightarrow \blacktriangle) \end{array}$$

Summary

Conditional and Biconditional Rules:

- **Modus Ponens** (**MP**), **Modus Tollens** (**MT**), and **Biconditional Elimination** (**↔ −**) are Elim rules, and should be used automatically whenever possible.
- **Biconditional Introduction** (**↔+**) is an Intro rule, and should be used only to yield a specific sentence needed to complete a deduction or to set up an Elim rule.

Deductions Involving Conditionals and Biconditionals:

- To **deduce a conditional**, automatically **use CD** (unless a simpler way of deducing the conditional is obvious).
- To **deduce a biconditional**, deduce the corresponding **conditional and converse** using CD, then **use Bicon Intro** (**↔+**) (unless a simpler way of deducing the biconditional is obvious).
- In an **Indirect Deduction** where the **negation of a conditional** is available, **use** that negation **as half of the** needed **contradiction**, and derive the other half (the conditional) using CD.

- **A conditional follows validly from** (i) the **negation of** its **antecedent**, and (ii) **from** its **consequent**.
- **The negation of a conditional entails** both (i) its **antecedent**, and (ii) the **negation of its consequent**.