2.20. Validity and Inconsistency

1. Validity and Inconsistency. Of the semantic concepts catalogued earlier, **inconsistency** turns out to have an especially close connection to **validity**.

Consider: if an argument is invalid, it has a **validity counterexample** – a valuation making all its premises true, and its conclusion false. But (from the semantic rule for negations) when the conclusion is false, the negation of the conclusion is true.

That suggests a new way of describing validity counterexamples.

> A **validity counterexample** is a possible situation (valuation) where **all the premises are true** and the **negation of the conclusion is true**.

So corresponding to a validity counterexample for an argument we have a "**counterexample set"** for that argument: the set of sentences containing the premises, and the negation of the conclusion.

> **Counterexample set** for an argument: the set
> {Premises, Negation of Conclusion}

In a validity counterexample for an argument, all the sentences in the counterexample set will be true – **simultaneously satisfied**. And since an invalid argument is just an argument with a validity counterexample, "counterexample set" provides a new definition of "invalid argument".

> **Validity counterexample** for an argument: a valuation simultaneously satisfying that argument's counterexample set.

> **Invalid argument:** an argument whose counterexample set is **consistent** (**simultaneously satisfiable**).

For instance, in our earlier invalid argument, Valuation 2 is a validity counterexample. And that valuation makes true all the sentences in the argument’s counterexample set, {(R ∨ S), R, ~S}.

(R ∨ S) . R ∴ S

	(2)		(1)	∴	
	R	S	**(R ∨ S)**	S	**~S**
	1	1	1	1	0
⇨	**1**	0	**1**	0	**1**
	0	1	1	1	0
	0	0	0	0	1

The argument is invalid precisely because its counterexample set is consistent (simultaneously satisfied).

Likewise a valid argument is just one with **no** validity counterexample. That means: an argument whose counterexample set is **not** simultaneously satisfiable – hence **inconsistent**. Inconsistency thus provides a novel definition of “valid argument” as well.

Valid argument: an argument whose counterexample set is **inconsistent**.

Compare our old contrast – whether an argument is valid or invalid – with our latest contrast – whether the counterexample set is consistent or inconsistent. We find the two distinctions line up perfectly.

	Valid Argument	**Invalid Argument**
Argument Valid?	YES	NO
Counterexample Set Inconsistent?	YES	NO

So with our earlier valid argument, no valuation simultaneously satisfies its counterexample set {(P ∨ Q), ~P, ~Q}.

(P ∨ Q) . ~P ∴ Q

		(1)	**(2)**	∴	
P	Q	**(P ∨ Q)**	**~P**	Q	**~Q**
1	1	1	**0**	1	**0**
1	0	1	**0**	0	1
0	1	1	1	1	**0**
0	0	**0**	1	0	1

The argument is valid precisely because {(P ∨ Q), ~P, ~Q} is inconsistent.

Understanding validity by way of inconsistency makes a certain intuitive sense. For we could say of a valid argument that, once we've accepted the premises, there's **no rational way to deny** the conclusion.

2. Logical Equivalence and Inconsistency. While validity can be understood in terms of inconsistency, **logical equivalence** can in turn be understood in terms of **validity**.

> Two sentences are **logically equivalent** just when each **follows validly** from the other.

So "P" and "~~P" are logically equivalent precisely because each follows validly from the other: whenever one is true, the other is true.

P	**~P**	**~~P**
1	0	**1**
0	1	**0**

Valid

P

∴ ~~P

Valid

~~P

∴ P

But the sentences “(P ∧ Q)” and “P” are *not* logically equivalent, precisely because we *don’t* find each following validly from the other: while “P” does indeed follow from “(P ∧ Q)” (as the first valuation shows), “(P ∧ Q)” does not follow validly from “P” (as the second valuation shows).

P	**Q**	**(P ∧ Q)**
1	1	1
1	0	**0**
0	1	0
0	0	0

Valid

(P ∧ Q)

∴ P

Invalid

P

∴ (P ∧ Q)

Applying our link between validity and inconsistency, we can rephrase this point about logical equivalence, like so.

> Two sentences ● and ▲ are **logically equivalent** just in case both {●, ~▲} and {~●, ▲} are **inconsistent**.

For every case of logical equivalence between two sentences brings two matching valid arguments; and we’ve seen that the validity of each of those arguments amount to their respective counterexample sets being inconsistent. Logical equivalence amounting to two valid arguments, it (equivalently) amounts to two counterexample sets each being inconsistent.

(More intuitively: if two sentences are logically equivalent, it’s logically impossible to have either one without the other.)

So, for example, “P” and “~~P” are logically equivalent because both {~P, ~~P} and {P, ~~~P} are inconsistent. Because the two sentences are equivalent, we can’t accept one as true while denying the other.

By contrast, “P” and “(P ∧ Q)” are not logically equivalent. And sure enough, we don’t here have the two inconsistent sets required of logical equivalence. For while {(P ∧ Q), ~P,} is indeed inconsistent, the set {P, ~(P ∧ Q)} is perfectly consistent – as seen from the second valuation, below.

P	Q	(P ∧ Q)	**~(P ∧ Q)**
1	1	1	0
1	1	0	**1**
0	0	0	1
0	0	0	1

Here again, our consistency-based measure makes sense. For if two sentences are logically equivalent, there should be no **possible** way of having one true without the other.

By retooling our definitions of validity (and related concepts) in terms of counterexample sets and consistency, we remove all reference to **valuations.** While that may seem a mere curiosity at this point, it will prove useful when adapting these concepts to the later **truth tree** method.[1]

[1] Note that reference is still made here to semantic concepts, because we also appeal to the concepts of consistency and inconsistency, which are here understood in terms of truth and falsehood. If we could provide non-semantic definitions of “consistency” and “inconsistency” – ones making no appeal to truth or falsehood – we could likewise provide entirely non-semantic definitions of “valid argument” and “logical equivalence”. This is explored further below (beginning in 2.34). in the discussion of proofs and deductions.

Summary: Validity and Inconsistency

- The **counterexample set** for an argument is the set {Premises, Negation of Conclusion}.
- An argument is **valid** if (and only if) its counterexample set is **inconsistent**.
- Two sentences ● and ▲ are **logically equivalent** just in case each **follows validly** from the other.
- Two sentences ● and ▲ are **logically equivalent** just in case the sets {●, ~▲} and {~●, ▲} are both **inconsistent**.