2.42. The Deductive Challenge: Proofs (and Deductions) without Derived Rules

The derived rule of De Morgan's Law streamlined and simplified the deductive apparatus so much that we could be forgiven for never wanting to go without it. Yet valuable lessons can found in using only the leaner, original set of rules – of somewhat the same sort learned by those undertaking survival training in the wilderness. For with practice a set of strategies emerge that are handy to know for any deductive purposes, and which moreover better reveal the nature of our deductive system. Deliberate adoption of such a 'no-frills' approach to deduction constitutes **The Deductive Challenge**.

1. Disjunctions. The most obvious piece of strategy comes from proofs (which lack any premises); for here Indirect Deduction is the only route open to us. And even for deductions (which have premises) it's already our practice to reach for Indirect Deduction automatically unless some obvious alternative presents itself.

So consider how we construct a proof for even so simple a theorem as T2.

Get: $(P \vee \sim P)$ (ID)

1.	$\sim(P \vee \sim P)$	AID

Without De Morgan's law to dispatch this inconvenient AID, it's not immediately clear what move to make.

But it's important to keep in mind that in an Indirect Deduction, if the argument is valid (or sentence really a theorem) there's bound to be a pair of contradictory sentences lurking inside the AID box. And in fact, as we noted earlier, if there's one such contradictory pair to be had – say, "P" and "~P" – then all the others are available as well.[1] So our immediate task is to decide which pair of contradictory sentences to get, and how to get them.

[1] As noted in 2.39.

A ‘brute force’ approach suggests itself here: construct two smaller IDs (within the main ID box), one for each of the two contradictory sentences – say, “P” and “~P”. But that would be a strategic mistake. For suppose we succeed in building a smaller ID for “P”, and then begin the ID for “~P”.

		Get: Theorem (ID)
1.	~Theorem	(AID)
		~~Get~~: P (ID)
3.	~P	AID
	(etc.)	
10.	P	3, 8, 9, ID
		~~Get~~: ~P (ID)
11.	~~P	AID

The second smaller ID for “~P” wins us as AID only “~~P” – a sentence already available, since we’d proven “P” earlier. Anything we get from “~~P” within the ID box we could have gotten without that ID box. So it’s never sensible strategy to show the two contradictory sentences in smaller, parallel IDs.

Instead of such an exhaustive sweep, a better approach is to show one of those two sentences, then survey the proof afterward to see what possibilities that sentence opens up.

Two points of strategy are helpful here. ***First***, it’s usually best to aim for the simplest of contradictory pairs: a **sentence letter and its negation**. ***Second***: it’s obviously strategic to pick a sentence letter that appears in a previous line – either in a premise or an AID.

So returning to our proof of T2, we first build a smaller ID to get “P”.

		Get: (P ∨ ~P) (ID)
1.	~(P ∨ ~P)	(AID)
		Get: P (ID)
2.	~P	AID
3.	(P ∨ ~P)	2, ∨+
4.	~(P ∨ ~P)	1, R
5.	P	2, 3, 4, ID

Now it’s clear that the same strategy would get “~P”; so it’s perfectly legal to write a “Get” line for “~P” and build another three-line ID here. But that ID box would be a waste of time, since the strategy we’d use inside that box can be executed on “P”, from Line 5.

		Get: (P ∨ ~P) (ID)
1.	~(P ∨ ~P)	(AID)
		Get: P (ID)
2.	~P	AID
3.	(P ∨ ~P)	2, ∨+
4.	~(P ∨ ~P)	1, R
5.	P	2, 3, 4, ID
6.	(P ∨ ~P)	5, ∨+
7.	(P ∨ ~P)	1, 3, 4, ID

This is a fairly general strategy for Chapter Two proofs: get one half of a contradictory pair of sentences – here, “P” on Line 5 – then see what new things that sentence allows us to deduce. For what we typically find is that the same strategy which worked inside the smaller ID (in this case, the one for “P”) can be repeated in the larger ID box.

That point accounts for the ‘echo chamber’ often encountered in proofs, as illustrated even in this simple example. Note that “$(P \vee \sim P)$” appears three times in seven lines: once as half of the contradiction in the smaller ID (Line 3), once as half the contradiction of the larger ID (Line 6), and finally as the sought-after sentence (Line 7).

		Get: $(P \vee \sim P)$ (ID)
1.	$\sim(P \vee \sim P)$	(AID)
		Get: P (ID)
2.	$\sim P$	AID
3.	$(P \vee \sim P)$	2, $\vee +$
4.	$\sim(P \vee \sim P)$	1, R
5.	P	2, 3, 4, ID
6.	$(P \vee \sim P)$	5, $\vee +$
7.	$(P \vee \sim P)$	1, 3, 4, ID

And finally this proof illustrates that $\vee +$ is our friend when it comes to deducing with a minimal toolbox. For while $\wedge +$ yields a conjunction only when we have both halves, $\vee +$ needs only one half of the disjunction to yield the whole sentence. In the Spartan climes of the unadorned proof, an inference rule that makes do with half-rations is especially welcome.

[2. Other Sentence Types. While our example here has been a proof of a disjunction – and though our listed theorems in this chapter have been almost without exception disjunctions – that doesn’t mean only disjunctions can be Chapter Two theorems. As a trivial example: for every disjunction that’s a theorem, its double negation will also be a theorem; and for any two disjunctions that are theorems, their conjunction is also a theorem.

Still, we can make some general observations on what sorts of sentences will be theorems in this formal language, and what strategy we should adopt to prove such a sentence.

It is first of all obvious that **no** basic – **sentence letter or negation of** a **sentence letter** – **could be a theorem**. For every basic is true in some valuation and false in some valuation, and so will be neither a theorem nor a contradiction.

Note also that since a conjunction is only true when both its parts are true, a conjunction can be a theorem only if both its parts are theorems. From our previous point that means that **no conjunction of basics could be a theorem**.

But it also means that any time we have a **conjunction** as a theorem we can prove each of its parts separately. Our proof strategy for conjunctions will thus be to prove each part of the conjunction individually, then conjoin them with ∧✚.

Similarly, for any **double negation** **~~●** that's a theorem ● must also be a theorem; so we prove ● and then derive **~~●** with **~✚**.

If a **disjunction** is a theorem we're not guaranteed that both parts are theorems (as we were with a conjunction). But we can be sure that either part follows validly from the negation of the other. For otherwise that disjunction could not be a theorem.

Consider: if one part of the disjunction, ●, fails to follow validly from the negation of the other part, ▲, then there's a validity counterexample for the argument **~▲** ∴ ● – a valuation where the premise **~▲** is true but the conclusion ● is false. But where **~▲** is true, ▲ must be false. So such a counterexample is a valuation where both parts of the disjunction, ● and ▲, are false – making the entire disjunction false in that valuation, thereby preventing that disjunction from being a theorem (a tautology).

(A special case of this is when at least one part of the disjunction is itself a theorem. For on the one hand a theorem (a tautology) follows validly from any sentence, and so in particular from the other half of the disjunction. And conversely that other part of the disjunction is bound to follow from the negation of a theorem; for the negation of a theorem is a contradiction, from which any and every sentence follows.)

That point suggests a deductive strategy for proving a theorem that's a disjunction: show it indirectly, using the AID – the negation of that disjunction – repeatedly as half of the contradictory pair of sentences. We know that we only need one part of the disjunction to prove the whole through $\lor$**+**. We can show that part (either part) through ID in turn, confident that the AID entails the other part of the disjunction – which will likewise secure the whole disjunction through $\lor$**+**.

[Example]

When the **negation of a conjunction** is a theorem, the conjunction itself must be a contradiction. In that case we show the theorem indirectly, getting an AID that’s the double negation of a contradiction – equivalently (through **~–**) a contradiction. This case is the dual of the previous one, where a disjunction is a tautology: while we’re not guaranteed that both parts of the conjunction will themselves be contradictions, we can be sure that either part of the conjunction entails the negation of the other.[2]

That might tempt us to shake out each half of the conjunction through $\land$**–**, then do a smaller ID of the negation of one of those two sentences. But strategically that smaller ID would be a waste of time, for the same reason we saw earlier: if we have each half of the conjunction (● $\land$ ▲) – ● and ▲ – then start a smaller ID for, say, ~▲, the AID will just be , ~~▲, which we already had in ▲. Anything we could have got in the smaller ID we could get just as easily without it.

We’re better off instead sticking with our earlier strategy: constructing a smaller ID for some basic that appears in an earlier line. For example the negated conjunction “$\sim((\sim P \lor \sim P) \land (P \lor (P \land Q))$” is a tautology, so there must a proof of this sentence through ID and the deductive rules. We start in the way just described.

	Get: $\sim((\sim P \lor \sim P) \land (P \lor (P \land Q)))$ (ID)	
1.	$\sim\sim((\sim P \lor \sim P) \land (P \lor (P \land Q)))$	(AID)
2.	$((\sim P \lor \sim P) \land (P \lor (P \land Q)))$	1, **~–**
3.	$(\sim P \lor \sim P)$	2, $\land$**–**
4.	$(P \lor (P \land Q))$	2, $\land$**–**

[2] For further discussion of both these points – disjunction as tautology and conjunction as contradiction – see 3.6.1 Problems C and D.

Now, instead of trying to get the negation of Line 3 or Line 4, we aim a smaller ID at getting “P” or “~P”. It’s equally simple to get either one. (Lines 5, 6, and 7 of the left proof are Lines 8, 9, and 10 of the right one, and vice versa.)

3.	$(\sim P \vee \sim P)$	2, $\wedge-$
4.	$(P \vee (P \wedge Q))$	2, $\wedge-$
		Get ~P
5.	$\sim\sim P$	AID
6.	P	5, $\sim-$
7.	$\sim P$	3, 5, $\vee-$
8.	$\sim P$	5, 7, ID
9.	$(P \wedge Q)$	4, 8, $\vee-$
10.	P	9, $\wedge-$
11.	$\sim((\sim P \vee \sim P) \wedge (P \vee (P \wedge Q)))$	1, 8, 10, ID

3.	$(\sim P \vee \sim P)$	2, $\wedge-$
4.	$(P \vee (P \wedge Q))$	2, $\wedge-$
		Get P
5.	$\sim P$	AID
6.	$(P \wedge Q)$	4, 5, $\vee-$
7.	P	6, $\wedge-$
8.	P	5, 7, ID
9.	$\sim\sim P$	8, $\sim+$
10.	$\sim P$	3, 9, $\vee-$
11.	$\sim((\sim P \vee \sim P) \wedge (P \vee (P \wedge Q)))$	1, 8, 10, ID

Finally, if the **negation of a disjunction** is a theorem, the disjunction itself is a contradiction. And in this case we’re guaranteed that both parts of the disjunction are themselves a contradiction.[3] [So…]]

[3] This is the dual of the earlier point that if a conjunction is a theorem then both its parts are theorems.