3.10. Connective Duality Revisited: Conditionals, Biconditionals, and More

Having expanded the formal language in this chapter to include arrows and bicons, we now seek to extend our earlier treatment of duality to cover these new connectives (and the sentences made from them). While truth tables for conditionals and biconditionals require no expansion of our treatment of semantic duality, determining the proper dual for each connective will lead us to introduce further new connectives to the current formal language.

1. Duals of Conditionals. Recall that we have two (parallel) types of duality: the **semantic dual** of a truth table, enacted by the True/False Swap (and inherited by any sentence taking that truth table); and the **connective dual** of a sentence, captured by the Connective Swap.[1] The parallel between these types of duality reflects the parallel between construction and semantics. Since every construction rule has a matching semantic rule, applying the True/False Swap to a semantic rule yields a corresponding duality holding for the connective introduced by the matching construction rule.

Defining Connective Swap in terms of the connective introduced by the construction rule, we concluded that the wedge is the connective dual of the vel, and that the tilde is its own (is a "self-dual") – as illustrated by the following table.[2]

$$\begin{array}{c|c} \vee & \wedge \end{array}$$
$$\sim$$

And connective duality can likewise apply duality to 'sub-languages' such as the {~, ∧} and {∧,∨} languages, and other formal languages such as DNF and CNF.

Since the general format of truth tables is the same as in the previous chapter – just supplemented with semantic rules for conditionals and biconditionals – the True/False Swap remains the same as before: given a truth table, we find its dual table by replacing each True in the original by False, and each False by True.

[1] As discussed in 2.33, and extended in 2.34.
[2] The central line is the 'axis of duality': like a line of reflection in a mirror image, each connective finds its dual reflected on the other side of the line. (The tilde is then in effect its own mirror image.)

Using the True/False Swap Method on the semantic rule for conditionals yields a dual sentence type true in only one valuation: where the antecedent of the original conditional is false but its consequent is true. (Once again, to avoid confusion, we depict the effect of this swap by using "True" and "False" for the new values, in place of the "1" and "0" in the original truth table.)

●	▲	(● → ▲)
1	1	1
1	0	0
0	1	1
0	0	1

●	▲	Dual of (● → ▲)
False	False	False
False	**True**	**True**
True	False	False
True	True	False

So, for instance, the conditional "(P → Q)" has as a dual a sentence only true when "P" is false and "Q" is true.

P	Q	(P→ Q)	Dual of (P→ Q)
1	1	1	0
1	0	0	0
0	1	1	1
0	0	1	0

Versed as we are in methods for matching truth tables with sentences, we know that a truth table with only one true valuation is matched by a **valuation sentence** – in this case, "(Q ∧ ~P)".[3]

P	Q	(P → Q)	~P	(Q ∧ ~P)
1	1	**1**	0	**0**
1	0	**0**	0	**0**
0	1	**1**	1	**1**
0	0	**1**	1	**0**

[3] As discussed in 2.26.

But as a guide to extending the Connective Swap to conditionals, this is unhelpful. For "(Q ∧ ~P)" uses two connectives; so no **one** connective is recommended here as the connective dual of the arrow.

We see that the Chapter Three language is strikingly different from the language of Chapter Two. For among the remarkable features of the Chapter Two language is the fact that each connective finds its dual within that language.[4] But that is not so for the current language. To extend the Connective Swap to conditionals, then, we must extend the formal language to include the **connective dual of the arrow** – a sentence whose semantic rule has the semantic properties of a conjunction with negated right part, like "(Q ∧ ~P)".

Now "(Q ∧ ~P)" is the formal translation of a "**without**" sentence such as "Rex passed Chemistry without studying" or "Neko swallowed her food without chewing it". So as the formal counterpart to "without" we introduce the connective "**%**" – whose resemblance to the abbreviation "**w/o**" should remind us of its meaning. To further stress this connection, we call the "**%**" symbol "**wo**".[5]

●	▲	(● % ▲)
1	1	0
1	0	1
0	1	0
0	0	0

Note that wo sentences, like conditionals, are sensitive to the order of their parts: just as "(P → Q)" and "(Q → P)" take different truth tables, so likewise do "(P **%** Q)" and "(Q **%** P)".

P	Q	(P % Q)	(Q % P)
1	1	**0**	**0**
1	0	**1**	**0**
0	1	**0**	**1**
0	0	**0**	**0**

[4] In technical parlance, the Chapter Two language is "closed under duality" using the Connective Swap. See 3.11 for more on closure and self-duality.

[5] The wo is sometimes referred to as "difference" (or "asymmetric difference").

That makes sense semantically: the English sentences "Rex passed Chemistry without studying" and "Rex studied without passing Chemistry" don't have the same meaning.

And the fact that order makes a difference affects logical duality. For recall that the dual of "(P → Q)" was not "(P **%** Q)," but "(Q **%** P)". **When taking the dual of a conditional via Connective Swap, the antecedent and consequent must switch places**. We add wo to our list as the Connective Swap **dual of the arrow**.

∨	∧
→	**%**

~

2. Duals of Biconditionals. The True/False Swap Method for "(P ↔ Q)" likewise yields a truth table had by no single connective in our current formal language. This is the truth table for a sentence only true when one or the other of its parts is true – but not when both are.

●	▲	(● ↔ ▲)
1	1	1
1	0	0
0	1	0
0	0	1

●	▲	Dual of (● ↔ ▲)
False	False	False
False	**True**	**True**
True	**False**	**True**
True	True	False

Our phrasing of those truth conditions calls to mind the **exclusive "or"**: "P or Q, but not both". And exclusive "or" does indeed fit the bill.

P	Q	(P ↔ Q)	(P ∨ Q)	(P ∧ Q)	~(P ∧ Q)	((P ∨ Q) ∧ ~(P ∧ Q))
1	1	**1**	1	1	0	**0**
1	0	**0**	1	0	1	**1**
0	1	**0**	1	0	1	**1**
0	0	**1**	1	0	1	**0**

Since our formal language lacks a single connective for such a sentence, we introduce the symbol “⊕” – called “**exor**” – to express an exclusive disjunction.[6] The **exor** is the **dual of the bicon** in the Connective Swap Method.

●	▲	(● ⊕ ▲)
1	1	0
1	0	1
0	1	1
0	0	0

The list of connective duals is extended accordingly.

$\vee$	$\wedge$
$\rightarrow$	**%**
$\leftrightarrow$	$\oplus$

~

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3. Further Connectives. In both of the above cases of duality, we already had a formal sentence taking the dual truth table in question – indeed, infinitely many such sentences. We introduced special connectives *only* so we could have a single-connective sentence associated with that truth table, in order to pair the arrow and bicon with another connective for purposes of Connective Swap.

But once we set our sights on matching each truth table with a single-connective sentence, we spot certain truth tables which are not yet so matched. We list here all the four-valuation truth tables.

[6] This is sometimes called “symmetric difference”.

?	?	(P → Q)	(Q → P)	(P ∨ Q)	(P ↔ Q)	~P	P	Q	~Q	(P ⊕ Q)	(P ∧ Q)	(P % Q)	(Q % P)	?	?
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
1	0	1	1	1	1	0	1	1	0	0	1	0	0	0	0
1	1	0	1	0	0	0	1	0	1	1	0	1	0	0	0
1	1	1	0	0	0	1	0	1	0	1	0	0	1	0	0
1	1	1	1	0	1	1	0	0	1	0	0	0	0	1	0

Truth Tables 1, 2, 15, and 16 are not yet associated with a single connective.

[It's true that if we restrict ourselves to one sentence letter, "(P → P)" will take Truth Table 1 and "(P % P)" will take 16. But in the interest of assigning a unique connective to each truth table, we add a new connective in each of the four remaining cases. <"P bicon P" likewise works, and takes semantic dual "P exor P".> <Rephrase in terms of semantic rule. Note also that we'll later want to explore languages that cover this truth table, but don't have an arrow or a bicon, or whatever.>]

Truth Table 1 is true regardless of what "P" and "Q" are; and likewise for the falsehood in Truth Table 16. <"(P → P)" is true regardless of what value "Q" takes – since "Q" doesn't appear in the sentence. But in fact "(P → P)" is likewise true regardless of what value "P" takes.> So a semantic rule (for a connective) matching Truth Table 1 needn't involve sentence letters at all; and likewise for a single connective taking Truth Table 16. Whereas a tilde is a **one-**place connective (yielding a complete sentence, with its own truth table, when one sentence is added), and all our other connectives are **two-**placed (two sentences short of a complete sentence), Tables 1 and 16 each correspond to a **logical constant** – a **zero-placed connective** with no blanks, hence needing no sentences added to take a truth table.

Truth Table 1 appears in the semantic rule of the logical constant "⊤" (pronounced "**tee**"). "⊤" is always true, regardless of what value "P" or "Q" take.

●	▲	⊤
1	1	1
1	0	1
0	1	1
0	0	1

To emphasize that sentence letter values play no role here, we could instead have written the semantic rule for "⊤" like this.

⊤
1

And to Truth Table 16 we assign the logical constant "⊥" – which, since it's an inverted "⊤", will take the inverted name "**eet**".[7]

●	▲	⊥
1	1	0
1	0	0
0	1	0
0	0	0

Truth Table 15 takes the "neither… nor" sentence "~(P ∨ Q)".

P	Q	**~(P ∨ Q)**
1	1	0
1	0	0
0	1	0
0	0	1

[7] Following Smullyan 1992 [Goedel's Incompleteness Theorems]: 132.

Matching this truth table will be the semantic rule for the two-placed connective "↓", called the "**dagger**"[8].

●	▲	**(P ↓ Q)**
1	1	0
1	0	0
0	1	0
0	0	1

Finally, Truth Table 2 takes the "not both" sentence "~(P ∧ Q)".

P	Q	**~(P ∧ Q)**
1	1	0
1	0	1
0	1	1
0	0	1

Corresponding to this truth table is the semantic rule for the two-placed connective "│", called the "**stroke**".[9]

●	▲	**(P │ Q)**
1	1	0
1	0	1
0	1	1
0	0	1

[8] This is sometimes called a "NOR". But the name "dagger" suggests an easy mnemonic for remembering its truth conditions: the **d**ownward **d**agger **d**enies a **d**isjunction.
[9] This is sometimes called a "NAND" (on analogy with "NOR").

We now have a single (zero- or one- or two-placed) connective assigned to each possible (four-valuation) truth table.[10]

$\top$	$(P \mid Q)$	$(P \rightarrow Q)$	$(Q \rightarrow P)$	$(P \vee Q)$	$(P \leftrightarrow Q)$	$\sim P$	P	Q	$\sim Q$	$(P \oplus Q)$	$(P \wedge Q)$	$(P \% Q)$	$(Q \% P)$	$(P \downarrow Q)$	$\bot$
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
1	0	1	1	1	1	0	1	1	0	0	1	0	0	0	0
1	1	0	1	0	0	0	1	0	1	1	0	1	0	0	0
1	1	1	0	0	0	1	0	1	0	1	0	0	1	0	0
1	1	1	1	0	1	1	0	0	1	0	0	0	0	1	0

In terms of duality, the True/False Swap Method makes clear that **the semantic dual of** “⊤” **is** “⊥” (and vice versa) – for a truth table with “1” in every valuation becomes a truth table with “0” in every valuation (and vice versa). So “⊤” and “⊥” act as **connective duals** of one another.

Likewise **the semantic dual of the dagger is the stroke**. For applying the True/False swap to the semantic rule for the dagger yields a sentence only false when both its parts are true – the “not both” truth table of the stroke.

●	▲	(● ↓ ▲)
1	1	0
1	0	0
0	1	0
0	0	1

●	▲	(● | ▲)
False	False	True
False	True	True
True	False	True
True	**True**	**False**

So the stroke serves as the **connective dual** of the dagger (and vice versa).

[10] If we liked we could replaced the entries for “P” and “Q” – which contain no connectives – by introducing a one-placed connective “**i**” such that “**i**●” is logically equivalent to “●”. “P” could then be replaced by “**i**P,” and “Q” by “**i**Q,” so that every entry contained a connective. Since the connective “**i**” is its own semantic dual, it can be treated as a connective self-dual as well.

We are therefore equipped with a master set of connectives – call it "**Formal Language A**" – where (i) every (four-valuation) truth table has a single-connective sentence, and where (ii) every connective finds its dual within that language.

$$\mathsf{A}\text{: } \{\,|\,, \rightarrow, \leftrightarrow, \vee, \top, \sim, \bot, \wedge, \oplus, \%, \downarrow\}$$

And the Connective Swap duality will follow this final table of duals.

$\vee$	$\wedge$
$\rightarrow$	$\%$
$\leftrightarrow$	$\oplus$
$\vert$	$\downarrow$
$\top$	$\bot$

$\sim$

We turn next to the effect these new connectives have on issues of **expressive adequacy**.