5.11. Models, Quantifiers, and Validity: Examples

With formal semantics in hand for names and predicate letters and quantifiers, we can at last tackle the real topic of interest in logic: the validity of arguments.

In sentence logic the informal notions surrounding validity were retooled in terms of **valuations** – so that a **validity counterexample** became a valuation where all the premises of the argument are true, but the conclusion false. Now that we have recast our semantics in terms of models, a validity counterexample will be a **model** where all the premises of the argument are true, but the conclusion false.

What follows are simple examples illustrating the mechanics of validity within this enhanced semantics.

Example 1. The following English argument appears intuitively **invalid**.

1. Some men are Americans.
2. Socrates is a man.

∴ Socrates is an American.

For a situation where only some men are American, and Socrates is among the non-American men, would be a validity counterexample for this argument.

Evaluating the argument for validity *formally* involves securing its logical form via translation, and testing that form semantically. Using the following translation key, the argument translates formally like so.

A: Socrates **H**: is an American
G: is a man

1. $\exists x\,(Gx \wedge Hx)$
2. GA

$\therefore$ HA

As in previous chapters, it is shrewd to test for validity **indirectly**: assuming a validity counterexample for the argument, then appealing to the formal semantics to assess the coherence of that assumption. So here we picture a validity counterexample, with both premises true and the conclusion false.

1 1. $\exists x\,(Gx \wedge Hx)$
1 2. GA

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0 $\therefore$ HA

Any model must have a certain minimal structure, which we can set out from the outset. First, every model must have a domain containing at least one object.

$\mathbb{D}$: {2}

Since we also require every object to have a name, we call Object 2 “A”.

$\mathbb{D}$: {2}

A: 2

This argument uses two predicates, “G” and “H”. Each predicate must have an extension in the model – though each such extension may be empty.

So we begin with that minimum commitment: an empty extension for each predicate letter.

$\mathbb{D}$: {2}

A: 2 **G**: { }
H: { }

From there we fill in further details of the alleged counterexample in light of the truth conditions for each sentence in the argument.

For instance, the second premise "GA" is true here. So the referent of "A" – Object 2 – must be in the extension of "G". Likewise the conclusion "HA" is false; so Object 2 must *not* be in the extension of "H".

1 1. ∃x (Gx ∧ Hx)
1 2. GA

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0 ∴ HA

$\mathbb{D}$: {2}

A: 2 **G**: {2}
H: { }

The first premise is a true existential sentence, requiring for its truth at least one true instance, obtained from its scope formula "(Gx ∧ Hx)" by replacing every free "x" with a name letter from the model.

The model so far uses only name letter "A". But the instance "(GA ∧ HA)" won't be true as the model stands; for with "GA" true and "HA" false, the conjunction "(GA ∧ HA)" will be false.

And it's no good trying to fix this by adding Object 2 to the extension of "H". While that would make "(GA ∧ HA)" true, it would do so only by making "HA" true. But then the conclusion of the argument would be true, disqualifying the model as a validity counterexample.

Instead we need to add another object to our model – say, Object 3 – with a corresponding name letter, "B".

$\mathbb{D}$: {2, 3}

A: 2	**G**: {2}
B: 3	**H**: { }

With name letter "B" added, the model yields another instance for Premise 1: "(GB ∧ HB)". For that conjunction to be true, both "GB" and "HB" must be true. So we add Object 3 to the extensions of "G" and "H"

1 1. ∃x (Gx ∧ Hx)
1 2. GA

---

0 ∴ HA

$\mathbb{D}$: {2, 3}

A: 2	**G**: {2, 3}
B: 3	**H**: {2}

Instances of "∃x (Gx ∧ Hx)":

(GA ∧ HA): **0**	(A/x)
(GB ∧ HB): **1**	(B/x)

With at least one true instance, the first premise is true. And since "A" has its referent in the extension of "G" but not of "H", the second premise is true, but the conclusion is false. This model thus qualifies as a validity counterexample for the argument, establishing the argument's **invalidity**.

Example 2. Consider next this simple English argument.

1. All men are mortal.

∴ Raphael is mortal.

Intuitively the argument seems invalid, since Raphael may not be a man.We translate this into formal language using the following translation key.

1. All men are mortal.

∴ Raphael is mortal.

A: Raphael **G**: is a man
H: is mortal

1. $\forall x\,(Gx \rightarrow Hx)$

∴ HA

Again we assume a validity counterexample for the argument.

1 1. $\forall x\,(Gx \rightarrow Hx)$

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0 ∴ HA

As always, we begin our model minimally, with at least one named object, and empty extensions for all the predicate letters in the argument.

$\mathbb{D}$: {2}

A: 2 **G**: { }
H: { }

Since “HA” is false, we leave Object 2 out of the extension of “H”. But do we need another object, to make the premise true?

In fact we don’t; for the model as it stands already makes that sentence true. Premise (1) is a universal sentence, true only if each of its instances in the model is true. Since “A” is the only name in the model, there is only one instance for Premise (1).

$\mathbb{D}$: {2}

A: 2 **G**: { }
H: { }

$(GA \rightarrow HA)$

Object 2 is not in the extension of “G” or “H,” rendering both “GA” and “HA” false; so the conditional “$(GA \rightarrow HA)$” is **true**.

Since all (one) of its instances are true in the model, the universal sentence “$\forall x\,(Gx \rightarrow Hx)$” is true as well. So the model already makes the premise true and the conclusion false – qualifying as a validity counterexample. No further objects are needed in its domain. (In a world populated only by the immortal angel Raphael, all men in that world would be mortal – all zero of them – but Raphael wouldn’t be.)

A Remark on Universals and Truth

While that last result might seem unintuitive, our semantics for universal sentences makes the following universal sentence true when there are no objects that are G.

$\forall x\,(Gx \rightarrow Hx)$

That is a direct consequence of the semantic rule for conditionals – specifically, that a conditional with a false antecedent is true. For with no object in the extension of “G”, any instance for “$(Gx \rightarrow Hx)$” is a conditional with a false antecedent, hence a *true* conditional.

By the same token, with no objects in the extension of "G" the following universal will also be true.

$\forall x\,(Gx \rightarrow {\sim}Hx)$

For again: the antecedent of each instance will be false, rendering every instance true – and so making the universal sentence true as well.

In the actual world where there are no unicorns, for example, our formal semantics counts *both* "All unicorns are pink" *and* "All unicorns are non-pink" as true in our world.

But the oddness of having both universals true is 'quarantined'. For the only time both

(1) $\forall x\,(Gx \rightarrow Hx)$

and

(2) $\forall x\,(Gx \rightarrow {\sim}Hx)$

are both true is when no object is G – that is, when the sentence

(3) ${\sim}\exists x\, Gx$

is also true.

In fact, we will show that Sentence (3) *follows validly* from (1) and (2). Likewise Sentences (1) and (2) follows validly from the sentence (3).

Example 3. Next we consider a slightly more complex English argument.

1. All doctors are university graduates.
2. Some doctors are women.

∴ Some women are university graduates.

The following translation key translates the argument formally like so.

1. All doctors are university graduates.
2. Some doctors are women.

∴ Some women are university graduates.

G: __ is a doctor
H: __ is a university graduate
I: __ is a woman

1. $\forall x\,(Gx \rightarrow Hx)$
2. $\exists x\,(Gx \wedge Ix)$

$\therefore\ \exists x\,(Ix \wedge Hx)$

We assume a validity counterexample.

1 1. $\forall x\,(Gx \rightarrow Hx)$
1 2. $\exists x\,(Gx \wedge Ix)$

---

0 $\therefore\ \exists x\,(Ix \wedge Hx)$

As usual we begin the model minimally, with one named object and an empty extension for each predicate letter in the argument.

$\mathbb{D}$: {2}

A: 2

G: { }
H: { }
I: { }

1 1. $\forall x\,(Gx \rightarrow Hx)$
1 2. $\exists x\,(Gx \wedge Ix)$

---

0 $\therefore\ \exists x\,(Ix \wedge Hx)$

Since the second premise is true, it must have at least one true instance. To keep our model as lean as possible, we try having Object 2 yield that true instance, "(GA ∧ IA)".

For that conjunction to be true, both its parts must be true. So Object 2 must be in the extension of both "G" and "I".

$\mathbb{D}$: {2}

A: 2

G: {2}
H: { }
I: {2}

The first premise "$\forall x\,(Gx \rightarrow Hx)$" is also true. For a universal sentence to be true in a model, every one of its instances must be true in the model. With "A" the only name in the model so far, "(GA → HA)" must be true.

Since Object 2 is already in the extension of "G," the antecedent of that conditional, "GA," is already true. The truth rule for conditionals tells us there's only one way to have the whole conditional and its antecedent true at the same time: when the consequent is also true.

●	▲	(● → ▲)
1	**1**	1
0	1	1
1	0	0
0	0	1

Hence the consequent "HA" must also be true. Object 2 must be in the extension of "H".

$\mathbb{D}$: {2}

A: 2	**G**: {2}
	H: {2}
	I: {2}

1 1. $\forall x\,(Gx \rightarrow Hx)$ ✓

1 2. $\exists x\,(Gx \wedge Ix)$ ✓

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0 $\therefore\ \exists x\,(Ix \wedge Hx)$

The conclusion "$\exists x\,(Ix \wedge Hx)$" is false, so every instance must be false. With only one name letter "A" in the model, there is only one such instance: "(IA $\wedge$ HA)". So "(IA $\wedge$ HA)" must be false.

But it's not possible for "(IA $\wedge$ HA)" to be false in this model. With Object 2 already in the extension of both "I" and "H," both "IA" and "HA" are true – making the conjunction "(IA $\wedge$ HA)" true as well. And with at least one true instance, the conclusion is then true.

So there's no **possible** way the premises of this argument could be true, while the conclusion is false. That is: if the premises are true, the conclusion must also be true. This argument is **valid**.

(Note: it's no use trying to add further objects to the domain of the model to yield a validity counterexample. For no matter how many objects we add, or what predicate extensions they fall in, some object must be in the extension of "G" and "I" to make Premise (2) true; and that object then has to be in the extensions of both "G" and "H" to keep Premise (1) true. But that suffices to make the conclusion true.)