3.6. Conditionals and Validity

1. Conditionals and Validity. Despite the linguistic and semantic innovations introduced in this chapter, conditions on validity remain unchanged: an argument is valid if (and only if) any valuation simultaneously satisfying the premises also satisfies the conclusion. But with conditionals in hand we can draw a connection that would not have been so obvious previously.

First, note that each argument has a corresponding sentence: a conditional with premises as antecedent, and conclusion as consequent.[1] (If the argument has more than one premise, the *conjunction* of these premises forms the antecedent.[2]) So the following valid argument has this "**leading principle**".[3]

1. If Rex's team lost, then Rex is upset.	1. $(P \rightarrow Q)$
2. Rex's team lost.	2. P
$\therefore$ Rex is upset.	$\therefore Q$

Leading Principle: $(\underline{(P \rightarrow Q) \wedge P}) \rightarrow Q)$

And the following invalid argument takes the accompanying 'leading principle'.

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[1] Technically: since we take each conditional (indeed, each formal sentence) to be only finitely long, only an argument with finitely many premises will have a leading principle. There is nothing objectionable in principle to an argument having an infinite number of premises; but such an argument would not have a leading principle.

[2] We formed a **conjunction** out of multiple premises precisely to guarantee this match between tautology and validity. For a valid argument, the conclusion must be true whenever **all the premises are true**; and in a validity counterexample the conclusion is false while **all the premises are true**. Since a conjunction is true only when **all its parts are true**, the conjoining of all the premises together is true when (and only when) all the premises are true.

[3] Following (Burgess 2009: 6); (Leonard 1957: 488) calls an argument's leading principle its "justifying principle".

1. If Rex's team lost, then Rex is upset.
2. Rex is upset.

∴ Rex's team lost.

1. $(P \rightarrow Q)$
2. Q

$\therefore P$

Leading Principle: $(\underline{(P \rightarrow Q) \wedge Q}) \rightarrow P)$

What makes that point interesting is the semantic profile of each such conditional. For the **valid** argument, its leading principle is a **tautology**.

$$\mathbf{(P \rightarrow Q)\ .\ P\ \therefore\ Q}$$

P	Q	$(P \rightarrow Q)$	$((P \rightarrow Q) \wedge P)$	$\mathbf{((P \rightarrow Q) \wedge P) \rightarrow Q)}$
1	1	1	1	**1**
1	0	0	0	**1**
0	1	1	0	**1**
0	0	1	0	**1**

And on reflection that should come as no surprise. A valid argument is one where no valuation makes the premises true yet conclusion false. But with premises serving as antecedent, and conclusion as consequent, this becomes: no valuation makes the antecedent true yet the consequent false. Since that is the one sort of valuation which makes a conditional false, our conditional is thus guaranteed to be false in no valuation – hence a tautology.

For the **invalid** argument, its leading principle is **not a tautology**.

$$\mathbf{(P \rightarrow Q)\ .\ Q\ \therefore\ P}$$

P	Q	$(P \rightarrow Q)$	$((P \rightarrow Q) \wedge Q)$	$\mathbf{((P \rightarrow Q) \wedge Q) \rightarrow P)}$
1	1	1	1	**1**
1	0	0	0	**1**
0	1	1	1	**0**
0	0	1	0	**1**

This too stands to reason: the argument was invalid because there is a validity counterexample – a valuation making all the premises true, yet the conclusion false. But that valuation likewise makes the antecedent of the conditional true and its consequent false – rendering the whole conditional **false** in that valuation. So the conditional is not a tautology.

This result holds in general.

> Each argument has a "**leading principle**": a conditional with the premise(s) of the argument (conjoined together) as antecedent, and conclusion of the argument as consequent.
>
> An argument is **valid** if (and only if) its leading principle is a **tautology**.

Despite this striking parallel between arguments and conditionals, however, it would be a mistake to view conditionals as arguments, or arguments as conditionals. For I do not stake the same claim(s) when asserting an argument as I do when asserting its leading principle. Asserting an argument will, in the bargain, assert both the premise(s) and the conclusion.

1. I won the lottery.

$\therefore$ I'm a millionaire.

The only way I can sincerely assert the above argument is by asserting that I won the lottery, and that I'm a millionaire.

But when I assert the leading principle of this argument I don't assert either of those claims.

If I won the lottery, then I'm a millionaire.

This conditional doesn't say that I won the lottery nor that I'm a millionaire – only that there's a link between the one event's holding and the other's.

So we continue to recognize a difference between arguments and conditionals; but we now also recognize a close link between the two.

2. Biconditionals and Logical Equivalence. Thanks to the link between validity and logical equivalence, our observations about leading principles can be extended to cover logical equivalence as well. Recall that when two sentences are **logically equivalent** (have the same truth table), each sentence **follows validly** from the other. For instance, "P" and "~~P" are logically equivalent; and each follows validly from the other.

P	**~P**	**~~P**
1	0	**1**
0	1	**0**

Valid

P

———

∴ ~~P

Valid

~~P

———

∴ P

Applying the above moral about leading principles, that means: the conditional counterparts of each argument is a **tautology**. Truth tables bear this out.

P	**~P**	**~~P**	**(P → ~~P)**	**(~~P → P)**
1	0	1	**1**	**1**
0	1	0	**1**	**1**

But since the second conditional is the converse of the first, the two conditionals together are equivalent to a **biconditional**. And indeed, the biconditional made from "P" and "~~P" is itself a tautology.

P	**~P**	**~~P**	**(P ↔ ~~P)**
1	0	1	**1**
0	1	0	**1**

This point holds in general.

> Two sentences are **logically equivalent** if (and only if) the **biconditional** built from those two sentences is a **tautology**.

And when two sentences are not logically equivalent, their corresponding biconditional is not a tautology. For instance, “(P ∧ Q)” and “P” aren’t logically equivalent; and their corresponding biconditional is not a tautology.

P	**Q**	**(P ∧ Q)**	**((P ∧ Q) ↔ P)**
1	1	1	1
1	0	**0**	**0**
0	1	0	1
0	0	0	1

3. Tautology and Consistency (Again). The above points provide a striking consolidation of our semantic concepts. For we originally treated as three separate matters (i) whether an argument is valid; (ii) whether two sentences are logically equivalent; and (iii) whether a sentence is a tautology. But with conditionals and biconditionals in hand, we see that the first two can be swept under the carpet of the third: testing a sentence for ‘tautology-hood’ by itself also serves as a test of validity or of logical equivalence. Somewhat surprisingly, perhaps, **being a tautology** appears to form the central concept of logic.

It would be more accurate, though, to say that introducing conditionals has instead provided a new way of thinking about some familiar semantic observations.

(I) We noted in the previous chapter[4] that the concept of **consistency** can be used to provide a new definition for "validity".

> **Counterexample Set** (for an argument): the set
> {Premises, Negation of Conclusion}

> **Valid argument**: an argument whose counterexample set is **inconsistent**.

But "counterexample set" was later translated into its sentence counterpart, the **counterexample sentence** for an argument.[5]

> **Counterexample Sentence** (for an argument): the conjunction of all the premises, and the negation of the conclusion, of that argument.

Once again the argument is valid if (and only if) its counterexample sentence is inconsistent – i.e., a contradiction. So the following argument is (again) valid.

$$\mathbf{(P \to Q)\ .\ P\ \therefore\ Q}$$

P	Q	~Q	$(P \to Q)$	$((P \to Q) \wedge P)$	$\mathbf{((P \to Q) \wedge P) \wedge \sim Q)}$
1	1	0	1	1	**0**
1	0	1	0	0	**0**
0	1	0	1	0	**0**
0	0	1	1	0	**0**

[4] In 2.17.
[5] In 2.28.

And this next argument is (again) invalid, since its counterexample sentence is consistent (satisfied in Valuation 3).

$$\mathbf{(P \rightarrow Q)\ .\ Q\ \therefore\ P}$$

P	Q	~P	(P → Q)	((P → Q) ∧ Q)	**((P → Q) ∧ Q) ∧ ~P)**
1	1	0	1	1	**0**
1	0	0	0	0	**0**
0	1	1	1	1	**1**
0	0	1	1	0	**0**

(II) Since the negation of a contradiction is itself a tautology, we can extend that last point: an **argument is valid** if (and only if) the **negation of its counterexample sentence** is a **tautology**.

But one more semantic observation brings these meditations full circle. Recall that a conditional is equivalent to the negation of a specific conjunction: the conjunction of the antecedent and negation of the consequent.[6]

"If Rex goes out, he'll take his umbrella": (P → Q)

"**It is not the case that** Rex will go out **without** taking his umbrella"
("Rex wo**n't** go out **without** taking his umbrella"): ~(P ∧ ~Q)

That means the negation of the counterexample sentence is likewise equivalent to a conditional: the conditional with the conjunction of argument's premises as antecedent, and conclusion of the argument as consequent.

But that's just the argument's leading principle again. Just as the argument is valid if (and only if) the negation of its counterexample sentence is a tautology, so **(III)** the argument is valid if (and only if) its leading principle is a tautology.

While the leading principle seemed at first to offer a new take on validity, we now recognize it as just a restatement of a familiar consistency-based approach.

[6] Noted in 3.3.

Summary: Conditionals and Validity, Biconditionals and Logical Equivalence

- Each argument has a **leading principle**: a conditional with the premise(s) of the argument (conjoined together) as antecedent, and conclusion of the argument as consequent.

- An argument is **valid** if (and only if) its leading principle is a **tautology**.

- Two sentences ● and ▲ are **logically equivalent** if (and only if) the biconditional (● $\leftrightarrow$ ▲) is a **tautology**.

- An argument's leading principle is equivalent to the negation of its **counterexample sentence**. So an argument's **leading principle is a tautology if (and only if)** its **counterexample sentence is a contradiction**.