3.11. Duality Meets Expressive Adequacy

1. Duals of Formal Languages. We seek next to extend duality, in both connective and semantic forms, to families of sentences and entire formal languages. There is some precedent for this in our earlier discussion: since each DNF sentence has as a connective dual a CNF sentence (and vice versa), the entire DNF language can be viewed as the dual of the CNF language. Here we work out the details of when one language is the dual of another, and what this reveals about the expressive adequacy of formal languages.

It's simple to extend Connective Swap duality to **sets of sentences**, and True/False Swap to **sets of truth tables**. For a set of sentences, its connective dual is just the set containing the connective dual of each sentence in the original set.

> **Connective Dual of Set-of-Sentences S**: the set containing just the connective dual of each sentence in set S.

Here's a simple example.

Set of sentences:	$\{(P \vee Q), \sim P, (\sim Q \wedge P)\}$
Connective dual of that set:	$\{(P \wedge Q), \sim P, (\sim Q \vee P)\}$

Sets of sentences are particularly interesting because a formal language can be viewed as just a set of sentences: all the sentences legal according to the construction rules of that language. With the concept of 'dual of a set of sentences' in hand we can speak meaningfully of **dual of a formal language**.

> The **dual of a formal language** is the language containing just the dual of each sentence in the original language.

But a second, more compact way of discussing a formal language, familiar from previous discussions[1], is to refer to the language by its set of connectives – for

[1] At the end of 2.27, and throughout 2.30.

example, referring to the Chapter Two language as the language {~, ∧, ∨}. This 'set of connectives' notation still picks out a formal language by its construction rules – taking for granted (both a construction rule providing sentence letters, and) that each connective listed follows its normal Chapter Two construction rule. We can then specify 'sub-languages' of the Chapter Two language by throwing out one or more connectives – for instance, the languages {~, ∧} and {∧, ∨}.[2]

So we can once again define the 'dual of a formal language' via connective duality.

> The **connective dual of a formal language L** is the language containing, as its connectives, just the dual of each connective in L.

For example, the language {~, ∧} has as its dual the language {~, ∨}.

And just as the tilde is a self-dual connective under Connective Swap duality, a formal language – treated as a set of connectives – counts as a connective self-dual if each connective in the set finds its dual connective within that set. For instance, the language {~, ∧**,** ∨} is a connective self-dual language, because the Connective Swap of each of these three connectives is found inside the set {~, ∧**,** ∨}. But the set {~, ∧} isn't a connective self-dual, because the dual of the wedge – namely, the vel – isn't found in the set {~, ∧}.

> A **language** is a connective **self-dual** if (and only if) each of its connectives finds its dual in that same language.

Connective self-duality applies likewise to our earlier treatment of a language as a set of sentences: if every sentence in the set finds its connective dual sentence in that set, the set counts as a connective self-dual. (Since the presence or absence of a sentence's connective dual in a language is strictly a matter of which connectives

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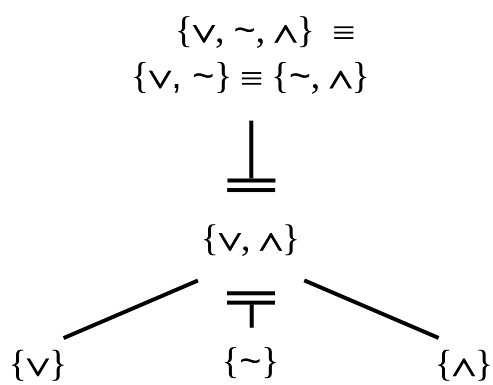
[2] Throwing out connectives isn't the only way of beginning with a language and pruning it down to some more austere language. The set of DNF sentences, for example, are only a subset of all the sentences in the {∧, ∨, ~} language, so DNF can in that sense be considered a 'sub-language' of the {∧, ∨, ~} language. Still, DNF uses all the connectives in the set {∧, ∨, ~}. (The same holds for CNF.) By contrast, in our discussion the 'sub-languages' of the Chapter Two language will be languages where connectives (ejected or not) follow the ordinary construction rules of Chapter Two – hence, e.g., with no special hierarchy of connective scopes, as in DNF or CNF.

the language has, the language-as-set-of-sentences treatment of self-duality is equivalent to the language-as-set-of-connective account.[3])

As a technical aside, note that self-duality of a set – whether of connectives, or of sentences – amounts to that set being **closed under duality**. A set of objects is said to be **closed** under a certain operation on those objects if the result of the operation never takes us outside that set. For instance, the set of positive integers {1, 2, 3…} is **closed under addition**, meaning: adding together any two numbers in that set yields as a result another number in that set. Addition within the set of positive integers never takes us outside that set.

If a set of connectives is **closed under duality**, each connective in the set finds its dual within that set; and a set of sentences is likewise closed under duality if each sentence in the set finds its dual sentence within that set. For a set of sentences or connectives, **closure under duality** is equivalent to being a **self-dual**.

In the following diagram[4] the connective self-dual languages are lined up along the middle (the "line of duality," or "line of reflection"). They are {∨, ~, ∧}, and sub-languages {∨, ∧}, and {~}.



[3] Again, assuming the connectives are treated in the language according to the ordinary construction rules of Chapter Two language. The DNF language, by contrast, introduces connectives via special construction rules (in 2.27 §3), so it would count as a (connective) self-dual in the set-of-connectives sense of "language," but not in the set-of-sentences sense. (A DNF sentence with all three connectives will have as its connective dual a CNF sentence not found anywhere in the set of DNF sentences.)

[4] From 2.30 §3 (and again leaving out the null language {} with no connectives, only sentence letters)

2. Duals of Truth Table Sets. The same treatment of set duality can be applied to **sets of truth tables** – though the duality in question will now be the semantic duality of the True/False Swap. Taking a truth table as any array of 2^N-many 1s and/or 0s (so: 2, or 4, or 8, or...), the **dual of a set of truth tables** will be the set containing just the semantic dual of each truth table in the original set.

For instance, Set T, containing the three truth tables A, B, and C, takes as its dual set the set D(T) containing truth tables D(A), D(B), and D(C).[5]

P	Q
1	1
1	0
0	1
0	0

Set T

A	**B**	**C**
0	1	1
0	0	1
0	0	1
0	0	1

Set D(T)

D(A)	**D(B)**	**D(C)**
1	1	0
1	1	0
1	1	0
1	0	0

(Note that D(A) is just truth table C by another name, and D(C) is just table A.)

Here again a set of truth tables is a **self-dual** if (and only if) each truth table in the set finds its semantic dual in that set. (In other words: if that set is **closed under** semantic **duality**.) So Set T, above, is not a self-dual; for while tables A and C each find their dual table within Set T (tables D(A) and D(C), respectively), B doesn't find its dual D(B) within Set T. By contrast, the smaller set of truth tables containing just A and C would be a self-dual.

And while we can discuss duality of arbitrary collections of truth tables, our real interest lies in the set of truth tables taken by the sentences of some formal language. The semantic rules pair each formal sentence with a matching truth table – and so, in general, pair the whole set of sentences of a language with the set

[5] In order to leave the truth tables for "P" and "Q" the same throughout, we used the Swap-and-Flip form of True/False Swap (from 2.33 §2) to get dual tables D(A), D(B), and D(C).

of truth tables which those sentences take. Call that the **truth table set** for the language.

> **Truth table set** for Language L: the set of truth tables taken by the sentences of Language L (according to the semantic rules)

Then a language is **expressively adequate** – providing some sentence to match any given truth table – if the truth table set for that language is just the **set of all possible truth tables**. Call that "**APT**" (for "all possible truth tables").

And likewise a language is **expressively inadequate** if there's some truth table which the language has no matching sentence for. (In other words: if the language's truth table set is a **proper subset** of APT.)

> A formal language is **expressively adequate** if it has APT as its truth table set; while a formal language is **expressively inadequate** if it doesn't have APT as its truth table set.

We noted before that each sentence in formal language L will find its connective dual sentence in the connective dual language D(L); so the sentences of L and D(L) pair up systematically via Connective Swap. But since True/False Swap always shadows Connective Swap, we also know that the truth table for a sentence in L takes as its semantic dual the truth table for that sentence's connective dual.[6] For instance: "(~P ∧ Q)" is a sentence of the language {~, ∧}, having as its connective dual the sentence "(~P ∨ Q)" from the dual language {~, ∨}. And the truth table for "(~P ∧ Q)" has as its semantic dual the truth table for "(~P ∨ Q)".

Now since this holds for every sentence in a formal language, we see that the truth table set for language L has as its dual the truth table set for connective dual language D(L) – each L truth table pairing up with its semantic dual D(L) table. That is: **just as formal languages pair up as duals, so do the truth table sets for those formal languages**.

[6] As discussed in 2.33 § 3.

That observation has an immediate consequence.

> **If a formal language is expressively adequate, so is its dual language**.

For to say that formal language L is expressively adequate is just to say that the language has APT as its truth table set. But since APT contains every possible truth table, APT is **closed** under semantic duality. (True/False Swap applies to every truth table, and each truth table obviously finds its dual within the set of all possible tables.) In other words: **APT is a self-dual** set of truth tables. So since language L has APT as its truth table set, the dual of language L also takes APT as its truth table set – which means that dual language is also expressively adequate.

The following claim is also true.

> **If a formal language is expressively inadequate, then so is its dual language.**

To say that language L is expressively inadequate means there's some truth table not found in L's truth table set. But since each truth table in L pairs up with a dual table from D(L), a truth table missing from the truth table set of L will have as its dual a truth table missing from the set for D(L).[7]

Letting a language inherit the semantic duality of its truth table set (just as a sentence inherits semantic duality from its truth table), we can define “semantic self-dual” for a language.

> A **language** is a **semantic self-dual** if (and only if) its truth table set is a semantic self-dual.

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[7] To put the argument indirectly: suppose for the sake of argument that **formal language L is expressively inadequate but its dual language D(L) is expressively adequate.** Since we've proven that an expressively adequate language has a dual language that's also expressively adequate, the dual of language D(L) must also be expressively adequate. But (because duality is involuntary) the dual of language D(L) is just L. So **L must be expressively adequate**, contradicting the original assumption that it's inadequate. Moral: there's no consistent way for a language to be expressively inadequate while its dual is expressively adequate.

Since a language and its connective dual have truth tables sets that are semantic duals, any language that's a connective **self-dual** has a truth table set that's a **semantic self-dual**. A point about sentence duality – that connective duality brings semantic duality in its wake – thus applies to languages as well.

But **a language can be a semantic self-dual without being a connective self-dual**.[8] For instance: the language {~, ∧} is expressively adequate, hence has as its truth table set APT, which is a self-dual; so {~, ∧} is a semantic self-dual. But {~, ∧} isn't a connective self-dual, since the dual connective of the wedge is the vel, which isn't found in {~, ∧}.

Finally, we return to the Tilde Insertion Method for constructing dual sentences discussed earlier.[9] Since, for any sentence, the Tilde Insertion Method yields a new sentence which is the (semantic) dual of the original sentence, any formal language containing the **tilde** (or its semantic equivalent) is bound to be a **(semantic) self-dual**.

> **Any language containing the tilde (or the semantic equivalent of the tilde) is a semantic self-dual.**

But we can't claim the converse – that any language which is a semantic self-dual contains the tilde (or its semantic equivalent). For instance, the language {∧, ∨} is (a connective self-dual, hence) a semantic self-dual. But {∧, ∨} doesn't contain the tilde; nor will any combination of wedges, vels, and sentence letters take the same truth table as, say, "~P".

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[8] So (predictably) the family of connective self-dual languages is a **proper subset** of the semantically self-dual languages.

[9] 2.34 § 3.

Summary: Duality and Expressive Power

- The Chapter Two language, and all its sublanguages, can be treated as the **set of sentences**, or (equivalently) as the **set of connectives** in the language.
- Treated as a set of sentences: the **dual of a language** is the set of connective duals of the sentences of that language.
- Treated as a set of connectives: the **dual of a language** is the set of connective duals of each connective in the original set.
- A **set of sentences** is a **self-dual** if each sentence in the set finds its connective dual sentence also in that set.
- A **set of connectives** is a **self-dual** if each connective in the set finds its connective dual connective also in that set.
- The **dual of a set of truth tables** is the set containing just the semantic dual of each truth table in that original set.
- A **set of truth tables** is a **self-dual** if each truth table in the set finds its semantic dual also in that set.
- The **truth table set** for a formal language is the set of truth tables taken by the sentences of that language.
- A **formal language is expressively adequate** if it takes as its truth table set the Set of All Possible Truth Tables (**APT**).

- If a formal language is **expressively adequate**, its dual language is also expressively adequate. If a language is **expressively inadequate**, its dual language is also expressively inadequate.

- To say that a formal language is a **semantic self-dual** is to say that its truth table set is a semantic self-dual.

- Any language that is a connective self-dual has a truth table set that is a semantic self-dual.

- If a **formal language contains the tilde** (or its semantic equivalent), then that language is a semantic self-dual. (But a language could be a semantic self-dual without containing the tilde or its semantic equivalent.)