3.10.1. Duality Problems

A. For each of the following formal sentences, state its **(connective) dual**.

1. $((P \rightarrow P) \rightarrow P)$
2. $(\top \rightarrow \bot)$
3. $(P \oplus \top)$
4. $(P \vee (Q \% P))$
5. $(P \oplus (P \leftrightarrow P))$
6.
7.
8.

B. Build truth tables for each sentence in (A). Which of these sentences is a **(semantic) self-dual**?

C. Use truth tables or truth trees to answer each of the following questions.

1. Decide if **wedge distributes over wo** – that is, decide if the following two sentences are **logically equivalent**.

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(a) $(P \wedge (Q \% R))$
(b) $((P \wedge Q) \% (P \wedge R))$

2. Decide if **vel distributes over wo** – that is, decide if the following two sentences are **logically equivalent**.

(a) $(P \vee (Q \% R))$
(b) $((P \vee Q) \% (P \vee R))$

3. Decide if **wedge distributes over exor** – that is, decide if the following two sentences are **logically equivalent**.

(a) $(P \wedge (Q \oplus R))$
(b) $((P \wedge Q) \oplus (P \wedge R))$

4. Decide if **vel distributes over exor** – that is, decide if the following two sentences are **logically equivalent**.

(a) $(P \vee (Q \oplus R))$
(b) $((P \vee Q) \oplus (P \vee R))$

5. Decide if **exor is associative** – that is, decide if the following two sentences are **logically equivalent**.

(a) $(P \oplus (Q \oplus R))$
(b) $((P \oplus Q) \oplus R)$

6. Does **bicon** have the same features as **exor**, from Problems (3) through (5)?

D. We treated an exclusive disjunction such as "P or Q, but not both" – which had previously been expressed as "$((P \vee Q) \wedge \sim(P \wedge Q))$" – as a much simpler sentence, by introducing a new connective in "$(P \oplus Q)$".

Suppose we do the same with an "otherwise" sentence. Currently the sentence "if P, Q; otherwise R" is translated as the conjunction of two conditionals.

If P, Q; otherwise R
$((P \rightarrow Q) \wedge (\sim P \rightarrow R))$

But let us now introduced a single three-place connective "#" to express such a sentence.

If P, Q; otherwise R
(P # Q R)

1. Show that the language {#, ⊤, ⊥} **is expressively adequate**.[1] *(Hint: use the adequacy of the {→, ~} language, by finding a {#, ⊤, ⊥} sentence equivalent to "~P," and one equivalent to "(P → Q)".)*

[1] Following the discussion in (Church 1956: 129-132) – though Church use the slightly different sentence form "$((Q \rightarrow P) \wedge (\sim Q \rightarrow R))$," which translates the sentence "P if Q; otherwise R".

2. Build a {#, ⊤, ⊥} sentence logically (semantically) equivalent to “(P ↔ Q)”.

3. Build a {#, ⊤, ⊥} sentence logically (semantically) equivalent to “(P ∧ Q)”.

C. Suppose we introduce a new connective, “**$**,” equivalent to the following.

> **Either not P and Q, or P and R**
> (P $ Q R)

1. Use **duality** to argue that the {**$**, ⊤, ⊥} **language is expressively adequate**.

2. Build a {**$**, ⊤, ⊥} sentence logically (semantically) equivalent to “~P”.

3. Build a {**$**, ⊤, ⊥} sentence logically (semantically) equivalent to “(P ∧ Q)”.

4. Build a {**$**, ⊤, ⊥} sentence logically (semantically) equivalent to “(P ⊕ Q)”.