3.12. Principles of Expressive Adequacy

We ended an earlier reading with a 'master set' of connectives, **A**.

A: { | , →, ↔, ∨, ⊤ , ~, ⊥ , ∧, ⊕, %, ↓}

Here we take up once more the issue of **expressively adequate** formal languages, and more generally whether each of these languages is **expressively equivalent** – yielding sentences which match the same set of truth tables. As before we treat a formal language as a set of connectives (taking for granted that the language also has sentence letters and parentheses). **A** is the largest formal language treated here, all of the languages discussed below forming 'sub-languages' of **A** (in the sense that they will contain only some of the connectives in **A**). Mapping these sub-languages illustrates important features of expressive power and duality.

1. Expressive Strength and Expressive Equivalence. Here we stop to spell out various assumptions about expressive power that were left intuitive and inexplicit in earlier discussions.

We will say that one language is **expressively at least as strong as** a second language, meaning: the sentences of the first language cover all the truth tables the second language covers – possibly more. For instance, language {∧, ∨, ~} is expressively at least as strong as language {∧, ∨}, since {∧, ∨, ~} covers all the truth tables covered by {∧, ∨} (and more – since {∧, ∨, ~} has a sentence matching the truth table for "~P," while, as we've seen, {∧, ∨} lacks any such sentence).

We use the "≥" symbol to express '(expressively) at least as strong as'. So we can restate our last observation as: {∧, ∨, ~} ≥ {∧, ∨}.

As a rule, a formal language will be at least as strong as any of its sub-languages (any smaller language got by throwing out one or more connectives from the original language). And we don't need to build any truth tables to see this, since *in general* throwing out connectives can't increase the number of sentences in that language, and so can't increase the number of truth tables which the language will have matching sentences for.

This is our first principle of expressive strength.

> **Principle I**: a formal language is expressively at least as strong as all of its sub-languages.

So all of the following claims hold from Principle I.

> 1. $\{\wedge, \vee, \sim\} \geq \{\wedge, \vee\}$
> 2. $\{\wedge, \vee, \sim\} \geq \{\wedge, \sim\}$
> 3. $\{\wedge, \vee, \sim\} \geq \{\vee, \sim\}$
> 4. $\{\wedge, \vee, \sim\} \geq \{\wedge\}$
> 5. $\{\wedge, \vee, \sim\} \geq \{\vee\}$
> 6. $\{\wedge, \vee, \sim\} \geq \{\sim\}$

Since **every language is expressively at least as strong as itself**, we can add a further (trivial) item to that list.

> 7. $\{\wedge, \vee, \sim\} \geq \{\wedge, \vee, \sim\}$

And *being expressively at least as strong as* is a **transitive** relation, meaning: if Language L1 is at least as strong as Language L2, and L2 is at least as strong as Language L3, then L1 is at least as strong as L3. For instance: $\{\wedge, \vee, \sim\} \geq \{\wedge, \vee\}$ (from 1); and $\{\wedge, \vee\} \geq \{\wedge\}$ (by Principle 1); so $\{\wedge, \vee, \sim\} \geq \{\wedge\}$.

Note further that **an expressively adequate language is at least as strong as every language**. For calling a language expressively adequate means that every possible truth table finds some matching sentence in that language. This is our second principle of expressive strength.

> **Principle II**: an expressively adequate formal language is at least as strong as *any* formal language.[1]

Since no language can match more than all possible truth tables, expressive adequacy is the high water mark of expressive strength.

DNF provided a general procedure for matching any truth table with a $\{\wedge, \vee, \sim\}$ sentence. But recall that we didn’t bother providing such a universal matching

[1] Keep in mind that “formal language” here means: Language A, or one of its sub-languages. (In a different sense – one going beyond the matching of truth tables – the formal language of Chapter Six is said to be ‘stronger’ than A or any of its sub-languages.)

procedure in order to show {$\wedge$, ~} and {$\vee$, ~} are adequate. Instead we simply showed that each of these languages is at least as strong as {$\wedge$, $\vee$, ~}. The assumption underlying that move forms our third principle of expressive strength.

> **Principle III**: If language L1 is expressively at least as strong as expressively adequate language L2, then L1 itself is expressively adequate.[2]

We can then understand other semantic relations in terms of being 'expressively at least as strong as'. So we say that two languages, L1 and L2, are **expressively equivalent** if (and only if) each language is at least as strong as the other. We use the symbol "≡" to state expressive equivalence.

L1 ≡ L2 if and only if: L1 ≥ L2 and L2 ≥ L1

Since we know that {$\wedge$, ~} is expressively adequate – and hence that {$\wedge$, $\vee$, ~} is also expressively adequate (by Principles I and III) – the two languages are expressively equivalent.

2. {$\wedge$, $\vee$, ~} ≥ {$\wedge$, ~}
8. {$\wedge$, ~} ≥ {$\wedge$, $\vee$, ~}

9. {$\wedge$, $\vee$, ~} ≡ {$\wedge$, ~}

We say that Language 1 is **expressively stronger than** Language 2 if (and only if): Language 1 is at least as strong as Language 2, but Language 2 is *not* at least as strong as Language 1. The symbol ">" will signify 'expressively stronger than'.

L1 > L2 if and only if: L1 ≥ L2 and L2 ≱ L1

For example, expressively adequate {$\wedge$, $\vee$, ~} is stronger than expressively inadequate {~}.

10. {$\wedge$, $\vee$, ~} > {~}

(In general: any expressively adequate language is stronger than any expressively inadequate one.)

[2] Likewise if L1 is expressively inadequate and L1 ≥ L2, then L2 is also expressively inadequate.

And we say that a **connective is redundant** in a certain formal language if throwing that connective out of the language leads to no loss of expressive strength (which truth tables get matching sentences). Put more technically: a connective is redundant in a certain language if (and only if) removing that connective yields a sub-language which is expressively equivalent to the original language. For example: since expressively adequate {∧, ∨, ~} has sub-language {∧, ~} which is also expressively adequate, "∨" is redundant in {∧, ∨, ~}. Likewise "∧" is redundant in {∧, ∨, ~}, since {∧, ∨, ~} ≡ {∨, ~}.[3]

We likewise say that **a language is redundant** if (and only if) that language contains a redundant connective. And we can say that **it would be redundant to add a certain connective** to a language if (and only if) the larger language yielded by adding the connective is expressively equivalent to the original langauge – that is, if adding the connective doesn't match any new truth tables with sentences. And

We next apply these concepts and principles to mapping the expressive strength of various formal languages.

[2. Insert here: section on one-connective adequate language]

2. Two-Connective Languages. Recall that when starting with the language {~, ∧, ∨, →, ↔}, a sub-language was adequate only if it contained a tilde. So even the sub-language {∧, ∨, →, ↔} was expressively inadequate, since it had no sentence matching the truth table for a negation such as "~P" or "~Q".

But with the influx of new connectives from the previous reading, we at last meet formal languages which lack the tilde, but still cover all possible truth tables.

[3] Note well: saying that "∧" and "∨" are each redundant in {∧, ∨, ~} means we could throw **either** connective out without loss of expressive strength. But it doesn't mean we could throw **both** out without loss of expressive strength; for the language that would result, {~}, is expressively inadequate – unlike original language {∧, ∨, ~}.

The language {→, ⊥}, for instance, can generate the truth table for a negation such as "~P"; for the sentence "(P → ⊥)" is equivalent to "~P".

P	⊥	(P → ⊥)
1	0	**0**
0	0	**1**

That establishes the following result.

11. {→, ⊥} ≥ {~}

From (11) we see that it would be redundant to add "~" to language {→, ⊥}.

12. {→, ⊥} ≡ {→, ⊥, ~}

Now, {→, ⊥, ~} is at least as strong as {→, ~} (by Principle I).

13. {→, ⊥, ~} ≥ {→, ~}

But we already know that {→, ~} is expressively adequate. So {→, ⊥, ~} is also expressively adequate (by 13 and Principle III). And since {→, ⊥} ≡ {→, ⊥, ~} (from 12), it follows that {→, ⊥} is expressively adequate.

14. {→, ⊥} is expressively adequate.

By contrast, we can show that language {↔, ⊥} is expressively inadequate. First, {↔, ~} has a sentence matching the "⊥" truth table: "(~P ↔ P)".

P	~P	(P ↔ ~P)
1	0	**0**
0	1	**0**

So it would be redundant to add "⊥" to language {↔, ~}.

15. {↔, ~, ⊥} ≡ {↔, ~}

Since "(P $\leftrightarrow \perp$)" is equivalent to "~P," it would likewise be redundant to add "~" to language {$\leftrightarrow$, $\perp$}.

P	$\perp$	(P $\leftrightarrow \perp$)
1	0	**0**
0	0	**1**

16. {$\leftrightarrow$, $\perp$, ~} $\equiv$ {$\leftrightarrow$, $\perp$}

So, since (by Principle II) {$\leftrightarrow$, ~} can (trivially) generate {$\leftrightarrow$}, {$\leftrightarrow$, ~} can generate {$\leftrightarrow$, $\perp$}.

6. {$\leftrightarrow$, ~} $\geq$ {$\leftrightarrow$, $\perp$}

And since {$\leftrightarrow$, $\perp$} $\geq$ {~} (from Result 2) it follows that{$\leftrightarrow$, $\perp$} $\geq$ {$\leftrightarrow$, ~}.

7. {$\leftrightarrow$, $\perp$} $\geq$ {$\leftrightarrow$, ~}

From (6) and (7), {$\leftrightarrow$, $\perp$} **and** {$\leftrightarrow$, ~} **are expressively equivalent**.

8. {$\leftrightarrow$, $\perp$} $\equiv$ {$\leftrightarrow$, ~}

But we already know that {$\leftrightarrow$, ~} is expressively inadequate. So {$\leftrightarrow$, $\perp$} is likewise expressively inadequate.

9. {$\leftrightarrow$, $\perp$} **is expressively inadequate**.

The next point follows straightforwardly from the fact that the dual (or: *a* dual) of any sentence can be stated by the Tilde Insertion Method.

> **Principle IV**: If it would be redundant to add "~" to a certain language, then it would also be redundant to add the dual of any connective in that language.

That holds trivially for {→, ⊥}: since it is adequate, it can generate everything, including the dual of {→}, {**%**}, and the dual of {⊥}, {**T**}. ({**T**} is generated by {→} alone.)

But the point also holds for the bicon. ({**↔**} by itself can generate {**T**}.)

10. {**↔**, ⊥} ≥ {⊕, **T**}

Moreover, "→" and "**↔**' can each generate {~} if combined instead with their dual (something which some connectives – e.g., "∨" or "**T**" – cannot do).

11. {→, **%**} ≥ {~}
12. {**↔**, ⊕} ≥ {~}

(11) entails that {→, **%**} is expressively adequate.

13. {→, **%**} **is expressively adequate**.

But (12) entails only (14).

14. {**↔**, ⊕} ≥ {**↔**, ~}

And since "~(P ↔ Q)" is equivalent to "(P ⊕ Q)," (15) follows.

15. {**↔**, ~} ≥ {**↔**, ⊕}

From (14) and (15), {**↔**, ~} and {**↔**, , ⊕} are expressively equivalent; and from (8), both are equivalent to {**↔**, ⊥}.

16. {**↔**, ⊕} ≡ {**↔**, ~} ≡ {**↔**, ⊥}

Note that the "**generation**" relation between languages **is preserved under duality** – as the following principle stresses.

> **Principle V**: If Language L1 generates L2, then the dual of L1 generates the dual of L2.

Example: Result (16) can be transformed into a new statement of expressive equivalence, thanks to duality. Performing a Connective Swap on (16) yields (17).

17. {⊕, ↔} ≡ {⊕, ~} ≡ {⊕, **T** }

(Note that {⊕, ↔} is a self-dual.)

Combining (16) and (17) yields the following equivalences.

18. {↔, ~} ≡ {↔, ⊥} ≡ {↔, ⊕} ≡ {⊕, ~} ≡ {⊕, **T** } ≡ {↔, ⊥, ~, **T**, ⊕}

{↔, ⊥, ~, **T**, ⊕} is one of only *three* five-connective languages that are expressively inadequate. It is its own dual.

Of course, if a language L is expressively adequate – generating the set of all connectives – then by Principle V, the dual of L generates the dual of the set of all connectives. But **the set of all connectives is its own dual** – for, containing all connectives, each connective within the set will find its Connective-Swap-dual also in that set. So Principle V yields the following as a special case.

> **Principle VI**: If a language is expressively adequate, then its dual is also expressively adequate.

An example: since {→, ⊥} is adequate, Principle VI entails that {**%**, **T**} is also expressively adequate.

19. {**%**, **T**} is **expressively adequate**.

Truth tables bear this out: the language {**%**, **T**} can generate the negation truth table.

●	▲	(● **%** ▲)
1	1	0
1	0	1
0	1	0
0	0	0

P	**T**	(**T** **%** P)
1	1	**0**
0	1	**1**

And since the negation of a *wo* sentence is equivalent to a conditional, this language also generates the conditional truth table. But {→, ~} is expressively adequate. Since {%, **T**} generates {→, ~}, {%, **T**} is also expressively adequate.

Two other two-connective languages bear note. First, since {⊕} alone generates {⊥}, {→, ⊕} generates {~} – as the following truth tables show.[4]

●	▲	(● ⊕ ▲)
1	1	0
1	0	1
0	1	1
0	0	0

P	(P ⊕ P)	(P → (P ⊕ P))
1	0	**0**
0	0	**1**

Since {→, ~} is expressively adequate, so is {→, ⊕}.

20. {→, ⊕} **is expressively adequate**.

By duality, (20) entails that {%, ↔} is also expressively adequate. Previous results show this to be true. For {↔} alone generates {**T**}; and {%, **T**} is expressively adequate (Result 19).

21. {%, ↔} **is expressively adequate**.

Finally, we saw earlier[5] that {→, ∨, ↔, ∧} is expressively inadequate. Since both {→} and {↔} generate {**T**}, {→, ∨, ↔, ∧} is equivalent to {→, ∨, **T**, ↔, ∧}. So {→, ∨, **T**, ↔, ∧} is expressively inadequate.

22. {→, ∨, **T**, ↔, ∧} **is expressively inadequate**.

That means its dual, {∨, ⊕, ⊥, ∧, %}, is also expressively inadequate.

23. {∨, ⊕, ⊥, ∧, %} **is expressively inadequate**.

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[4] We could prove this point without appeal to either truth tables or tildes. For we know (Result 4) that {→, ⊥} is expressively adequate. So since {→, ⊕} ≥ {→, ⊥}, by Principle III {→, ⊕} is expressively adequate.
[5] In 3.7.

Moreover, we noted that {→**,** ∨**,** **T**, ↔, ∧} is equivalent to four of its sub-sets: {→, ↔}, {∨, ↔}, {∧, ↔}, and {∧, →}.

24. {→, ↔} ≡ {∨, ↔} ≡ {∧, ↔} ≡ {∧, →} ≡ {→**,** ∨**,** **T**, ↔, ∧}

The dual of each of these smaller sets is likewise equivalent to the dual of the larger set.

25. {**%**, ⊕} ≡ {∧, ⊕} ≡ {∨, ⊕} ≡ {∨, **%**} ≡ {∨, ⊕, ⊥, ∧, **%**}

As it turns out, {→**,** ∨**,** **T**, ↔, ∧}, {↔, ⊥, ~, **T**, ⊕}, and {∨, ⊕, ⊥, ∧, %} are the **only** 5-connective sub-languages of **A** which are expressively inadequate. All other 5-connective sets, and all sub-languages of **A** with 6 or more connectives, are expressively adequate.

These three inadequate languages are all **redundant**. The largest non-redundant inadequate language is the four-connective set {∨, **T**, ⊥, ∧}.

2. Three-Connective Languages. Among the three-connective sub-languages of **A**, six (non-redundant) languages are expressively adequate.[6]

First, {∨, ↔, ⊥,} is expressively adequate. For by Result 2 {↔, ⊥,} ≥ {~}, so {∨, ↔, ⊥,} ≥ {∨, ~}. And {∨, ~} was shown in Chapter Three to be expressively adequate.

26. {∨, ↔, ⊥} **is expressively adequate**.

Second, {∨, ⊕, **T**} is expressively adequate. For {⊕, **T**} ≥ {~} (from Result 18), so {∨, ⊕, **T**} ≥ {∨, ~}; and {∨, ~} is expressively adequate.

27. {∨, ⊕, **T**} **is expressively adequate**.

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[6] Our strategy with each of these cases is to first show that the language generates {~}, then show that one of the connectives, combined with the tilde, forms an expressively adequate language.

Third, $\{\vee, \leftrightarrow, \oplus\}$ is expressively adequate, since (from 12) $\{\leftrightarrow, \oplus\} \geq \{\sim\}$.

28. $\{\vee, \leftrightarrow, \oplus\}$ **is expressively adequate**.

The remaining three languages are proven adequate from 26-28, by duality.

$\{\vee, \leftrightarrow, \bot\}$	$\{\top, \oplus, \wedge\}$
$\{\vee, \oplus, \top\}$	$\{\bot, \leftrightarrow, \wedge\}$
$\{\vee, \leftrightarrow, \oplus\}$	$\{\leftrightarrow, \oplus, \wedge\}$

29. $\{\top, \oplus, \wedge\}$, $\{\bot, \leftrightarrow, \wedge\}$, and $\{\leftrightarrow, \oplus, \wedge\}$ **are** all **expressively adequate**.

All the remaining 3-connective sub-languages of **A** are either expressively inadequate or redundant (possibly both).

3. One-Connective Languages. We have so far neglected mention of our new connectives stroke and dagger. These turn out to form the only expressively adequate **single-connective** sub-languages of **A**.

The dagger alone can generate $\{\sim\}$.

●	▲	(● ↓ ▲)
1	1	0
1	0	0
0	1	0
0	0	1

P	(P ↓ P)
1	0
0	1

And the negation of a dagger sentence is equivalent to a disjunction.

P	Q	(P ↓ Q)	((P ↓ Q) ↓ (P ↓ Q))
1	1	0	1
1	0	0	1
0	1	0	1
0	0	1	0

Since $\{\downarrow\} \geq \{\vee, \sim\}$, and $\{\vee, \sim\}$ is expressively adequate, $\{\downarrow\}$ is expressively adequate.

30. $\{\downarrow\}$ **is expressively adequate**.

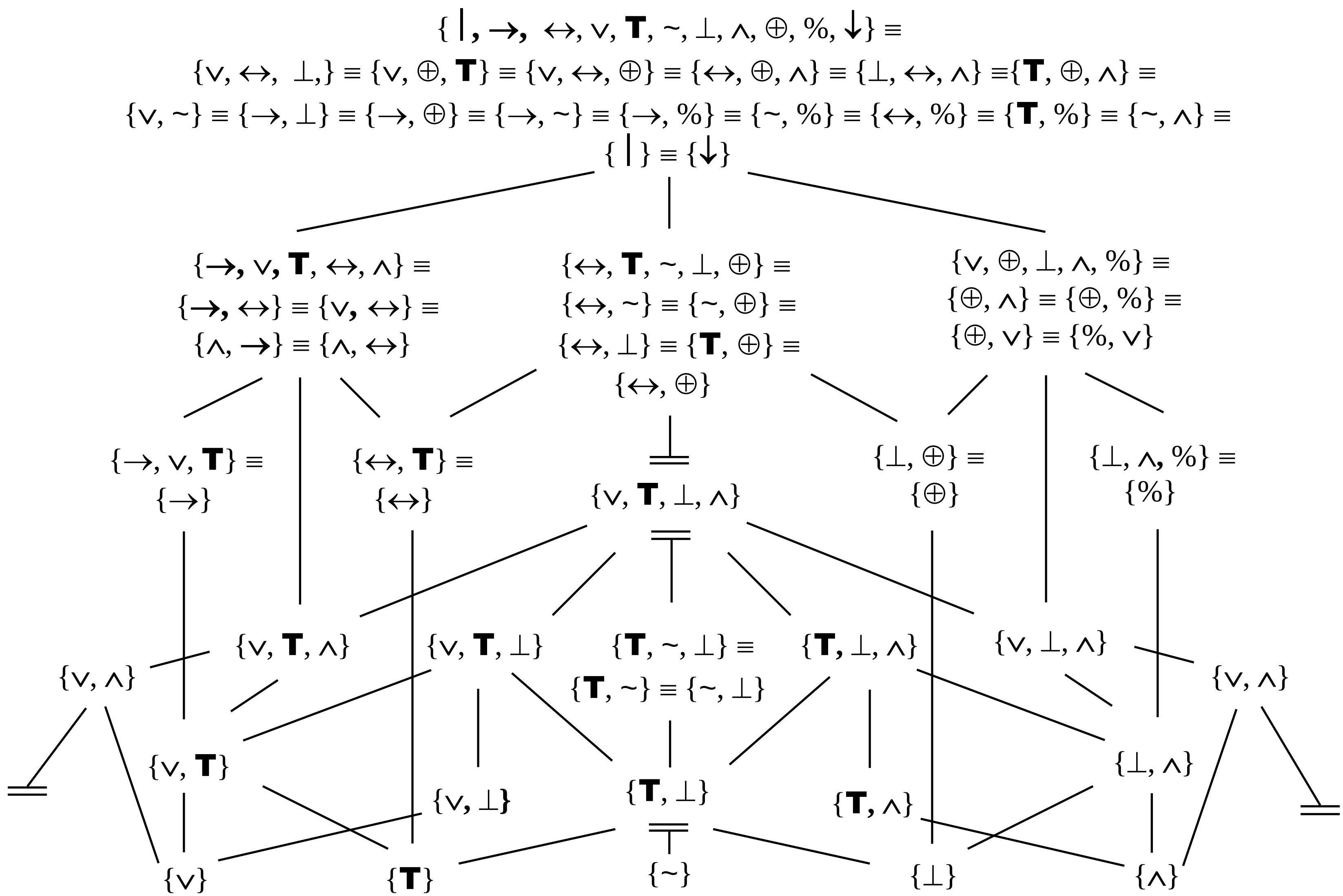
By duality, $\{\,|\,\}$ is also expressively adequate.

31. $\{\,|\,\}$ **is expressively adequate**.

Summary: Adequate, Non-Redundant Languages

A: {$|$, $\rightarrow$, $\leftrightarrow$, $\vee$, $\top$, $\sim$, $\bot$, $\wedge$, $\oplus$, %, $\downarrow$}

{$\downarrow$} {$|$}
{$\rightarrow$, %}
{$\rightarrow$, $\sim$} {$\sim$, %}
{$\rightarrow$, $\oplus$} {$\leftrightarrow$, %}
{$\rightarrow$, $\bot$} {$\top$, %}
{$\vee$, $\sim$} {$\sim$, $\wedge$}
{$\vee$, $\leftrightarrow$, $\oplus$} {$\leftrightarrow$, $\oplus$, $\wedge$}
{$\vee$, $\top$, $\oplus$} {$\leftrightarrow$, $\bot$, $\wedge$}
{$\wedge$, $\top$, $\oplus$} {$\leftrightarrow$, $\bot$, $\vee$}



Summary: A Family of Formal Languages