2.40. Derived Rules of Inference

While the deductive system so far developed will reliably provide a deduction for all (and only) the valid arguments in our formal language, many of these deductions are extremely long. Here we develop a method of deductive ‘shortcuts’ to reduce the size and complexity of our deductions.

1. Derived Rules. We established the validity of the following little argument by constructing an indirect deduction of it.

1. ~P

∴ ~(P ∧ Q)

1.	P	
		~~Get~~ ~(P ∧ Q) (ID)
2.	~~(P ∧ Q)	AID
3.	(P ∧ Q)	2, ~−
4.	P	3, ∧−
5.	~P	1, R
6.	~(P ∧ Q)	2, 4, 5, ID

Now this general argument form reappears in larger arguments such as the following.

1. (R ∧ ~P)

∴ (R ∧ ~(P ∧ Q))

Certainly we can construct a deduction of the conclusion from the premise. But doing so involve simply disassembling the conjunction on Line 1 using ∧−; deducing “~(P ∧ Q)” from “~P” in an ID; and then assembling the conclusion “(R ∧ ~(P ∧ Q))” from its two parts, via ∧+.

We are, in effect, simply pasting our earlier deduction of "~(P ∧ Q)" from "~P" into the middle of this larger deduction.

1.	(R ∧ ~P)	
		Get: (R ∧ ~(P ∧ Q))
2.	R	1, ∧–
3.	~P	1, ∧–
		Get: ~(P ∧ Q) (ID)
4.	~~(P ∧ Q)	AID
5.	(P ∧ Q)	4, ~–
6.	P	5, ∧–
7.	~P	3, R
8.	~(P ∧ Q)	4, 6, 7, ID
9.	(R ∧ ~(P ∧ Q))	2, 8, ∧+

But we could bypass this repeated labor by instead treating the valid argument "~P ∴ ~(P ∧ Q)" as an additional **rule of inference**. More precisely: we accept the following general argument **form** as a rule of inference.

Negation-Conjunction (~∧)

$$\frac{\sim\bullet}{\sim(\bullet \wedge \blacktriangle)}$$

In that case the deduction is simplified, as follows.

1.	(R ∧ ~P)	
		Get: (R ∧ ~(P ∧ Q))
2.	R	1, ∧−
3.	~P	1, ∧−
4.	~(P ∧ Q)	3, ~∧
5.	(R ∧ ~(P ∧ Q))	2, 4, ∧+

We're confident that using this rule will never compromise the validity of a deduction, since we can always instead paste in the deduction of the conclusion from the premise, as we did in our original deduction of "(R ∧ ~(P ∧ Q))" from "(R ∧ ~P)". Such a rule – added to the system, and justified by a deduction – is called a **derived rule** (in contrast with the **basic rules** built into the deductive system, which are 'basic' precisely because they're not justified by a deduction)[1]. Since the system is capable of all the same deductions without them, derived rules are not essential parts of the deductive system – just convenient shortcuts.

Suppose we call our original system of deduction (without Rule ~∧) **System 2** ("2" for the chapter in which it's presented). And let **System 2.1** be the system just like System 2 but also containing ~∧ as one of its **basic** (non-derived) rules. Whenever System 2.1 invokes its rule ~∧ in a deduction, System 2 can paste in its deduction matching that rule. So the same set of arguments are recognized as valid by both systems – the only difference being that the System 2 deduction will be longer, when pasting in the ~∧ deduction is necessary.

We will say that two systems are **deductively equivalent** when they pick out exactly the same arguments as valid. Being deductively equivalent to System 2, System 2.1 has no advantage except convenience: where an inference of the ~∧ form is involved, the System 2.1 deduction will be shorter. On the other hand, from the perspective of System 2 the inference rules in System 2.1 contain excess baggage – since throwing the rule ~∧ overboard brings no loss of deductive power.

[1] Of course, in justifying the derived rule by a deduction, that deduction must not appeal to the very rule being justified, or the justification will be circular. So in the above deduction of "~(P ∧ Q)" from "~P," we were careful not to appeal to the rule ~∧ anywhere in the deduction.

A slightly leaner third system – call it “**System 2.2**” – offers a further illustration of deductive equivalence. System 2.2 is like our System 2, except that it **has the $\sim\wedge$ rule and lacks our $\wedge-$ rule**. It might seem that certain valid arguments would escape the grasp of System 2.2 deductions– most obviously arguments of the following sort.

$$\frac{1.\ (P \wedge Q)}{\therefore\ P}$$

But that is not so. For System 2.2 has a deduction of this argument using only its basic inference rules.

1.	$(P \wedge Q)$	
		Get: P (ID)
2.	$\sim P$	AID
3.	$\sim(P \wedge Q)$	2, $\sim\wedge$
4.	$(P \wedge Q)$	1, R
5.	P	2, 3, 4, ID

Systems 2 and 2.2 are thus **deductively equivalent**. (A user of System 2.2 could, if she wished, treat our rule $\wedge-$ as a derived rule.) The existence of different, yet equivalent, deductive systems shows that we have some latitude in which deductive system we use to pick out the valid arguments. In this respect choice of deductive system is similar to our earlier choice among expressively equivalent formal languages.[2] In both cases, systems with quite different basic elements nonetheless prove equivalent.

2. De Morgan’s Law. Our point in discussing the rule $\sim\wedge$ was only to illustrate the concept of a derived rule. We won’t bother adding $\sim\wedge$ to our deductive system – making the judgment call that the convenience it brings is insufficient to justify complicating our list of inference rules.

[2] In 2.30; and later in 3.9 through 3.12.

But a different inference rule is useful enough to merit addition as a derived rule: **De Morgan's Law**.

De Morgan's Law (DM)

Inward DM | **Outward DM**

$$\frac{\sim(\bullet \vee \blacktriangle)}{(\sim\bullet \wedge \sim\blacktriangle)} \qquad \frac{\sim(\bullet \wedge \blacktriangle)}{(\sim\bullet \vee \sim\blacktriangle)} \qquad\qquad \frac{(\sim\bullet \wedge \sim\blacktriangle)}{\sim(\bullet \vee \blacktriangle)} \qquad \frac{(\sim\bullet \vee \sim\blacktriangle)}{\sim(\bullet \wedge \blacktriangle)}$$

We encountered these four valid argument forms earlier as semantic equivalences.[3] But they now serve as two types of inference rule: **inward DM**, which pushes a tilde into the parts of a disjunction or conjunction; and **outward DM**, which 'extracts' a tilde from the parts of a disjunction or conjunction. (We bother to label the two varieties of DM because they play different roles in deduction.)

Most obviously: inward DM allows for the easy dispatch of otherwise vexing AIDs, such as the following.

1. $(\sim P \vee R)$
2. $(\sim Q \vee S)$
3. $(P \vee Q)$

Get: $(R \vee S)$ (ID)

4. | $\sim(R \vee S)$ AID

Armed only with the seven deductive rules and ID, the situation looks bleak. The only move open to us here is to start a second ID within the first.

[3] In 2.17 § 1.

But with DeMorgan’s Law that fearsome AID is immediately tamed.

1.	$(\sim P \vee R)$	
2.	$(\sim Q \vee S)$	
3.	$(P \vee Q)$	
		Get: $(R \vee S)$ (ID)
4.	$\sim(R \vee S)$	AID
5.	$(\sim R \wedge \sim S)$	4, In DM

What follows is a thoroughly automatic cascade of Elim rules, backing its way into a contradiction.

1.	$(\sim P \vee R)$	
2.	$(\sim Q \vee S)$	
3.	$(P \vee Q)$	
		Get: $(R \vee S)$ (ID)
4.	$\sim(R \vee S)$	AID
5.	$(\sim R \wedge \sim S)$	4, In DM
6.	$\sim R$	5, $\wedge-$
7.	$\sim S$	5, $\wedge-$
8.	$\sim Q$	2, 7, $\vee-$
9.	P	3, 8, $\vee-$
10.	$\sim P$	1, 6, $\vee-$
11.	$(R \vee S)$	4, 8, 9, ID

Of course, to use De Morgan’s Law as a legitimate derived rule we must supply deductions establishing that the conclusion is indeed deducible from the premise in each case. Here is the deduction of the form just used.[4]

1.	~(P ∨ Q)	
	Get: (~P ∧ ~Q)	(ID)
2.	~(~P ∧ ~Q)	AID
	Get: ~P	(ID)
3.	~~P	AID
4.	P	3, ~−
5.	(P ∨ Q)	4, ∨+
6.	~(P ∨ Q)	1, R
7.	~P	3, 5, 6, ID
	Get: ~Q	(ID)
8.	~~Q	AID
9.	Q	8, ~−
10.	(P ∨ Q)	9, ∨+
11.	~(P ∨ Q)	1, R
12.	~Q	8, 10, 11, ID
13.	(~P ∧ ~Q)	7, 12, ∧+
14.	~(P ∧ Q)	2, 13, ID

With DM added to our deductive system we’re in a position to simplify the negation of any molecular sentence.[5]

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[4] Note that since Line 2 is not cited in the justification of Lines 3 through 13, we deduced Line 13 without using Line 2. We could thus have avoided using ID to deduce “(~P ∧ ~Q),” instead proceeding directly to the smaller IDs for “~P” and “~Q”. In that case Line 13 would be the last line of the deduction, and the large ID box (with its AID Line 2) would not appear – shaving two lines from the deduction.

[5] The negation of a negation is already handled by the rule ~−.

3. Deductive Strategy, Revised. In terms of strategy, Inward DM is of particular use in making an AID manageable. Whenever an ID begins with a negated conjunction or negated disjunction as its assumption, we now automatically apply inward DM to yield a disjunction or conjunction susceptible to Elim rules. For instance, the AID “~(P ∧ Q)” becomes “(~P ∨ ~Q)” (and ∨– is then applied if possible); while the AID “~(P ∨ Q)” becomes “(~P ∧ ~Q)” (with ∧– then applied).

Of course Inward DM proves handy for sentences other than an AID. In general, inward DM acts like an Elim rule – in the sense that it cannot be applied an unlimited number of times, and so can safely be executed whenever possible.

Outward DM can also only be applied a finite number of times, and so can also be trusted not to run amok. Still, our strategy will be to employ outward DM primarily as a ‘setup’ rule, using it in the same spots, and for the same reasons, as the Intro rules. For Outward DM leaves us with a sentence – a negated conjunction or negated disjunction – to which an Elim rule won’t automatically apply. So we use Outward DM chiefly to get a missing sentence needed when the deduction has ground to a halt: to complete an instance of ∨–; or to get half of a contradiction (when using ID) or the sentence on the “Get” line (when not using ID).

The addition of DM to the deductive system streamlines deductions substantially – so much so that adding further derived rules won’t prove necessary. Our system of Chapter 3 deduction has thus reached its finished form.

Summary: DeMorgan's Law Strategy

Inward DM (In DM)

$$\frac{\sim(\bullet \vee \blacktriangle)}{(\sim\bullet \wedge \sim\blacktriangle)} \qquad \frac{\sim(\bullet \wedge \blacktriangle)}{(\sim\bullet \vee \sim\blacktriangle)}$$

Outward DM (Out DM)

$$\frac{(\sim\bullet \wedge \sim\blacktriangle)}{\sim(\bullet \vee \blacktriangle)} \qquad \frac{(\sim\bullet \vee \sim\blacktriangle)}{\sim(\bullet \wedge \blacktriangle)}$$

- Treat **Inward DM** like an Elim rule: use whenever possible. In particular: automatically apply inward DM when the AID is a negated conjunction or negated disjunction.

- Treat **Outward DM** like an Intro rule: to supply (i) the missing part to apply an Elim rule; (ii) half of a contradiction (when using an ID); or (iii) the sentence on the "Get" line (when not using an ID).