5.12. Quantified Predicate Logic: Truth Trees

We adapt our earlier truth tree rules to the expanded formal semantics.

We've added only two new aspects to the semantics of previous chapters: a truth rule for predicate-letter-plus-name letter, and truth rules for universal and existential generalizations.

A predicate-letter-plus-name-letter is, in terms of truth and falsehood, an *atom* (since its 'sub-atomic' parts – predicate letter and name letter – aren't themselves true or false). And for that reason these sentences pose no new challenges for truth trees. They're like the other semantic atoms – the sentence letters – in not being further broken down in a truth tree.

For universal and existential generalizations, truth comes by way of instances, which act as semantic stand-ins for the scope formula. Based on the truth value of the instances, the quantifier yields a value for the whole quantified sentence. So while the truth value of a quantified sentence does depend on the truth value of a smaller sentence, that smaller sentence is not, technically, a *part* of that larger whole. (It's the scope formula which is part of a quantified sentence – not its stand-in, the instance.)

That will mark something unprecedented in truth trees. For truth tree rules have so far 'un-built' a sentence the same way a construction tree would – only with truth values added. But, following the semantic developed so far, truth tree rules for existential and universal generalizations will not break such a sentence into its immediate part (the scope formula), but instead into close relatives of that sub-part (one or more instances).

That innovation poses a new challenge: since there is no end to the number of instances we can build for a quantified sentence, truth trees threaten to spin out of control unless the rules for quantified sentences are constrained. To ensure a finite test, truth tree rules will need to focus on just those instances useful either to constructing a counterexample for the argument, or to showing that no such counterexample exists.

1. True Existentials, False Universals. Consider first a **true existential** sentence such as "Something is made of steel," translated into the formal language as follows.

"Something is made of steel."

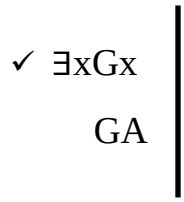
G: is made of steel.

$\exists xGx$

For this existential sentence to be true, there must be at least one true instance of its scope formula "Gx" – for example, "GA".

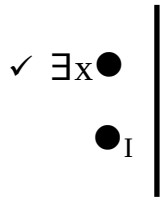
Instance of Gx: **GA** (A/x)

Our truth tree rule for true existentials reflects this.



In general: if an existential sentence is true, some instance of it is true.

True Existential *(First Draft)*



(Here $\bullet_I$ is an instance of the scope formula ●.)

But we must restrict this rule to avoid mistaking invalid arguments for valid ones. The following argument, for instance, is glaringly **invalid**.

1. Something is made of steel.

∴ Rex is made of steel.

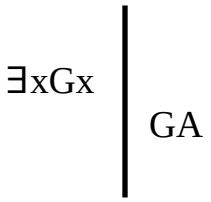
Adapting the previous translation key, we translate like so.

A: Rex
G: is made of steel.

1. ∃xGx

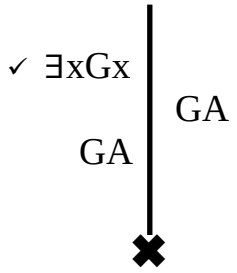
∴ GA

The tree test begins as usual: premises on the left, conclusion on the right.



But if we now build an instance "GA," the tree will close – because the atom "GA" is on both the left and right of the line.

☠ A Bad Tree ☠



In closing every line, the tree judges the original argument to be **valid** – clearly the **wrong** result.

Our misstep came in using the instance "GA" – in English, "Rex is made of steel". The true existential sentence "Something is made of steel" certainly promised us that some-object-or-other is made of steel. But it was a bold and invalid leap to assume that the object in question was Rex.

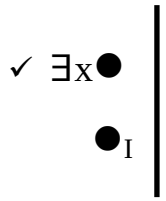
The same sort of mistake is illustrated in the following passage.

> Someone broke in to the museum between midnight and 2 AM, and stole the crown jewels. Let's call that person "Rex". So then: Rex broke into the museum and stole the jewels. Quick – grab Rex before he gets away!

While it's fine to call the thief "Mr. X," or some other name that no one is using, the name "Rex" was already taken; so it was far from innocent to use that name for the burglar.

To avoid this sort of error, we insist that with a true existential the name used in its instance can't *already* be used by something in the model – or that tree path. For true existential sentences the name used in the instance must be **new** to that path – i.e., a name not appearing previously on that path of the tree.

True Existential

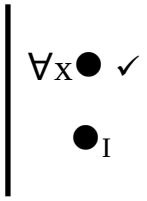


(*Restriction:* ●I *must use a* new *name letter*)

False universal sentences present a parallel case. With a *true* universal, all of its instances must be true. And that means even one false instance suffices for a false universal sentence.

So: a false universal sentence is guaranteed to have at least one false instance – as the tree rule recognizes.

False Universal *(First Draft)*



But again we had better restrict the name in that instance, if we don't want the tree test to yield mistaken judgments of validity. The following English argument is clearly **invalid**.

1. The Cathedral of Learning is made of limestone.

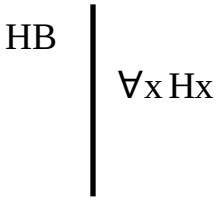
∴ Everything is made of limestone.

B: The Cathedral of Learning
H: is made of limestone.

1. HB

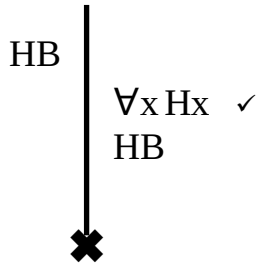
∴ ∀x Hx

Testing the formal argument for validity begins as usual.



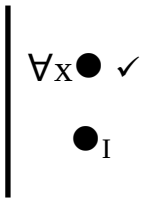
But the false instance "HB" leaves that atom on both the left and right sides of the line – hence marking the argument *valid*.

☠ A Bad Tree ☠



Again the error lay in allowing the instance to use a name *already* appearing on that tree path. We therefore impose the same restriction on the False Universal Rule.

False Universal



(*Restriction:* ●I *must use a* new *name letter*)

2. True Universals, False Existentials: 'Star Rules'. Compared with the truth tree rule for the false universal, the rule for a **true universal** sentence is quite simple. A true universal requires a **true instance for every name** used **in the model**.

So for this model, featuring name letters "A," "B," and "C," the true universal "∀x Gx" brings in its wake a true instance for each.

𝔻: {2, 3, 4}

A: 2
B: 3
C: 4

G: {2, 3, 4}

Instances of "∀x Gx":

GA: **1** (A/x)
GB: **1** (B/x)
GC: **1** (C/x)

That means that, unlike the previous tree rules for quantified sentences, the rule for a true universal **won't** be finished with just one instance. And while we used a three-name model here for illustration, in building a tree we won't know *in advance* how many objects (or names) might be needed for a validity counterexample.

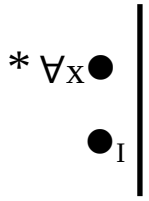
Indeed, in the course of a truth tree for an invalid argument we *back our way into* the appropriate model – just as, in previous chapters, we backed into a valuation which qualified as a counterexample. Instead of beginning with a model (and then constructing the appropriate instances) as we did in the above example, in a truth tree we have the model only when the tree is completed. And without a ready-made model to appeal to, we won't know in the course of the tree how many instances are called for.

Yet it would be madness for the tree rule to require *every possible* instance. For with an infinite number of name letters in the formal language, an infinite number of instances are possible for a quantified sentence. A tree rule requiring an infinite number of steps creates a truth tree test that *never* ends, for any (finite) number of steps – and that's no test at all.

True universals thus pull us in opposite directions: potentially requiring any number of instances, yet also requiring a cap on that number to keep the test from spinning out of control.

We address the first requirement with an addition to our bookkeeping notation. Whereas all previous tree rules *checked* a molecular sentence, and never returned to it, when extracting a true instance from a true universal we instead **star** the universal sentence – thereby noting that we may return to it later for further instances.

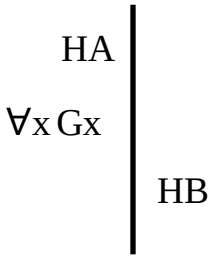
True Universal



We cap the potential explosion of instances with two restrictions.

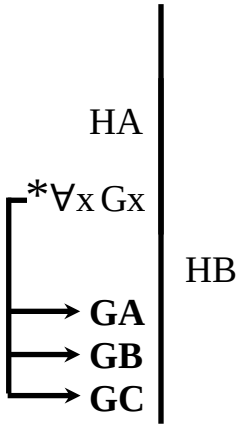
First, so long as there are already one or more name letters on that tree path, the True Universal rule builds instances **only** for those names. That is: the True Universal rule will **never introduce new name letters** (provided there's already a name letter on that path). Whereas we applied the True Existential and False Universal rules only to *new* name letters, here we do basically the opposite: whenever possible applying the True Universal rule only to '*old*' name letters.

In the following tree, with name letters "A" and "B" already appearing, we build two instances for "∀x Gx": "GA" and "GB".



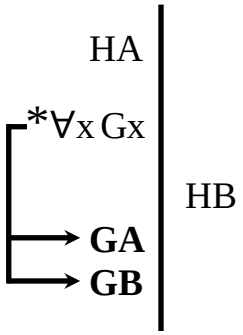
So in these three applications of the True Universal rule, the third is improper, since it builds an instance for *new* name letter "C".

☠ A Bad Tree ☠



We will, however, apply the True Universal Rule to 'old' (i.e. pre-existing) names **whenever** possible – specifically, to **every** 'old' name letter on that path. Since two name letters, "A" and "B," were already present in that last tree, it would have been equally improper to build *only* "GA" with the True Universal rule; for in leaving out instance "GB" we overlook 'old' name "B".

The proper application of the True Universal Rule in this case is as follows.



Second, in applying truth tree rules we push the True Universal rule to the back of the line. That is: we perform such a 'star' rule (a rule *starring* its sentence, rather than *checking* it) only **after** completing all the available 'check' rules. That way, any name letters to be added by check rules (True Existential, False Universal) will already be on the scene.

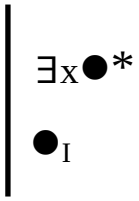
In practice those two restrictions work like so: beginning with a tree that may already contain name letters, we milk out as many more name letters as possible with the True Existential and False Universal Rules. Only *after* that do we apply the True Universal rule – building an instance for each of the names already found on that path of the tree.

The rule for **false existential** sentences is the mirror image of the True Universal rule. Since an existential sentence is true so long as even one instance is true, it is false only if *every* one of its instances is false.

Again we face a potential infinity of instances (here, false ones): if the sentence "Something is a unicorn" is false, then also false will be the sentences "Neko is a unicorn," "Jack is a unicorn," and so on. The consequences are the same as with a true universal. First: no one instance exhausts the potential of a false existential. Second: any attempt to build *all* possible instances is doomed to infinite regress.

Our solution is the same as well. In recognition of its unlimited potential, a false existential sentence is starred, not checked – leaving open the possibility of revisiting it later for further instances.

False Existential



Once again we apply this 'star' rule only *after* all available 'check' rules are used – seeking, whenever possible, to build an instance for (all and) *only* the names already present on that tree path.

A final point completes our treatment of the 'star' rules. We said earlier that 'star' rules apply only to 'old' names **so long as there are already (one or more) name letters on that line of the tree**. We appreciate the importance of that final proviso ("*so long as*…") when considering a case where *no* names appear, and *only* 'star' sentences remain to be broken down. The following **valid** argument provides an example.

1. Nothing is a unicorn.

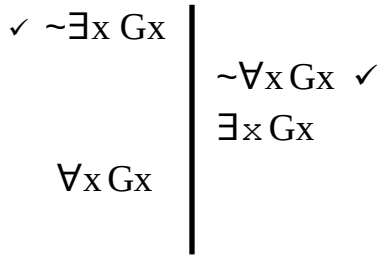
∴ Not everything is a unicorn.

G: is a unicorn

1. ~∃x Gx

∴ ~∀x Gx

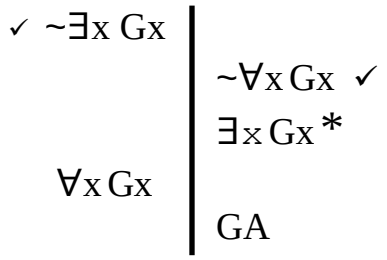
The truth tree test of validity begins as always; and we can easily dispatch the tildes in "~∃x Gx" and "~∀x Gx" with the True Negation and False Negation rules.



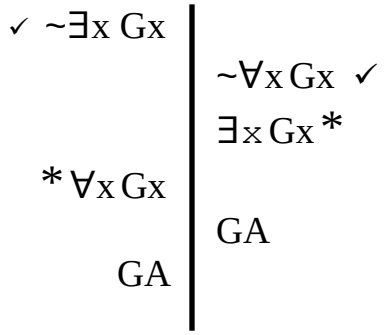
But then we hit a wall. Since false existentials and true universals are 'star' sentences, we seek to build instances of them using only names *already present* on that tree path. Yet no names are present, and no 'check' rules can be applied to supply a name. Counting the tree as finished, however, would wrongly judge the argument *invalid*.

We need a single name here to break the logjam. So we add: if, in breaking break down a 'star' sentence, there are *no* name letters present on that path of the tree, nor further 'check' sentences to supply names, we allow the 'star' sentence to introduce **one** new name letter as an "ice-breaking maneuver.

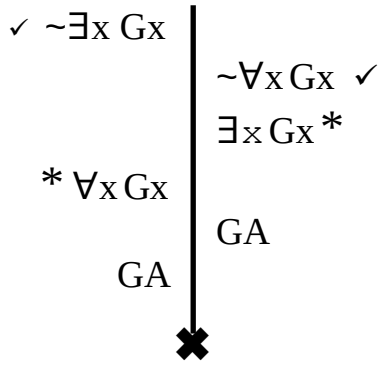
Here, for example, we allow "∃x Gx" to introduce an instance by the False Existential rule. (But then: no further *new* name letter may be introduced by 'star' sentences on this tree line.)



The True Universal rule then obligatorily applies "∀x Gx" to "A".



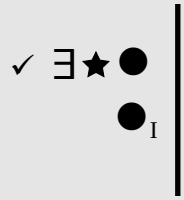
With "GA" on both sides of the line, the tree closes – correctly judging the argument valid.



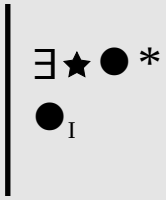
With this last-resort, 'ice-breaking' policy in place, our account of the "star" rules is complete.

Summary: Quantifier Truth Tree Rules

True Existential

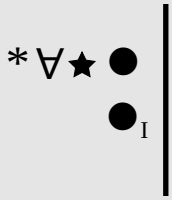


False Existential

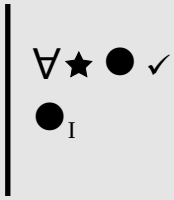


(*Restriction: ●I must use a **new** name letter*)

True Universal



False Universal



(*Restriction: ●I must use a **new** name letter*)

- 'Star' rules apply to a line **after** all 'check' rules for quantifiers have been made.
- A 'star' sentence makes instances to all and only name letters **already** on its line of the tree.

  (***'Ice-breaking' Exception***: if there are *no* name letters on the tree line, nor any 'check' rules on that line to introduce name letters, a 'star' rule can introduce **one** new name letter.)