

❖ *Formal Semantics: Further Issues* ❖

3.8. Expressive Adequacy Revisited: Conditional and Biconditional Languages:

We earlier explored issues of expressive power and expressive adequacy for the Chapter Two formal language $\{\sim, \wedge, \vee\}$, and various of its ‘sub-languages’. But with the advent of the arrow and biconditional sign (“bicon”) in the Chapter Three language, the issue of expressive adequacy rises again.

The full Chapter Three language $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$ is bound to be **expressively adequate** – capable of supplying a sentence to match any given truth table. For we established already that the Chapter Two language $\{\sim, \wedge, \vee\}$ is expressively adequate. But every $\{\sim, \wedge, \vee\}$ sentence qualifies as a $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$ sentence. So any possible truth table will be matched by some $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$ sentence – making the $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$ language expressively adequate.¹

More interesting is the question whether there are any expressively adequate **sub-languages** of $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$ which feature arrow or bicon.

In fact there are. And the simplest of these is the $\{\sim, \rightarrow\}$ language.

To prove $\{\sim, \rightarrow\}$ expressively adequate we use the same strategy applied earlier to the $\{\sim, \wedge\}$ language. Recall that, having established that $\{\sim, \wedge, \vee\}$ is expressively adequate, we showed that $\{\sim, \wedge\}$ sentences can generate any truth table which $\{\sim, \wedge, \vee\}$ sentences can – making $\{\sim, \wedge\}$ adequate as well. Since $\{\sim, \wedge\}$ is the same as $\{\sim, \wedge, \vee\}$ but for lack of \vee , the trick was to find a $\{\sim, \wedge\}$ form semantically equivalent to a disjunction. Finding such a form – “ $\sim(\sim\bullet \wedge \sim\blacktriangle)$ ” in place of “ $(\bullet \vee \blacktriangle)$ ” – settled that $\{\sim, \wedge\}$ could build any truth table which

¹ In general: adding further connectives to a formal language can only increase its expressive power – the set of truth tables covered by the sentences of that language.

$\{\sim, \wedge, \vee\}$ could. And a similar strategy established the adequacy of the $\{\sim, \vee\}$ language.

Likewise, if we show that $\{\sim, \rightarrow\}$ is expressively equivalent to an expressively adequate language, we settle that $\{\sim, \rightarrow\}$ is itself expressively adequate.

We achieve that end by constructing a $\{\sim, \rightarrow\}$ sentence form semantically equivalent to the conjunction. “ $\sim(\bullet \rightarrow \sim\blacktriangle)$,” will always take the same truth table as “ $(\bullet \wedge \blacktriangle)$ ”.

\bullet	\blacktriangle	$\sim\blacktriangle$	$(\bullet \rightarrow \sim\blacktriangle)$	$\sim(\bullet \rightarrow \sim\blacktriangle)$	$(\bullet \wedge \blacktriangle)$
1	1	0	0	1	1
1	0	1	1	0	0
0	1	0	1	0	0
0	0	1	1	0	0

$\{\sim, \rightarrow\}$ sentences can therefore cover all of the truth tables which $\{\sim, \wedge\}$ sentences can. But since $\{\sim, \wedge\}$ is expressively adequate, its sentences cover **all** possible truth tables. So $\{\sim, \rightarrow\}$ sentences do as well –establishing that $\{\sim, \rightarrow\}$ is **expressively adequate**.

And for the reasons rehearsed above, any larger language containing arrow and tilde will be expressively adequate as well – for example, $\{\sim, \rightarrow, \leftrightarrow\}$.

Indeed, any adequate sub-language of $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$ must contain a tilde, and either a wedge, vel, or arrow.

So these are the adequate sub-languages of $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$.

$\{\sim, \wedge, \vee, \rightarrow\}$	$\{\sim, \wedge, \vee\}$	$\{\sim, \wedge\}$
$\{\sim, \wedge, \vee, \leftrightarrow\}$	$\{\sim, \wedge, \rightarrow\}$	$\{\sim, \vee\}$
$\{\sim, \wedge, \rightarrow, \leftrightarrow\}$	$\{\sim, \vee, \rightarrow\}$	$\{\sim, \rightarrow\}$
$\{\sim, \vee, \rightarrow, \leftrightarrow\}$	$\{\sim, \wedge, \leftrightarrow\}$	
	$\{\sim, \vee, \leftrightarrow\}$	
	$\{\sim, \rightarrow, \leftrightarrow\}$	

We show that all remaining sub-languages of $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$ are expressively **inadequate**.

Of the single-connective languages, $\{\sim\}$, $\{\wedge\}$, and $\{\vee\}$ were proven inadequate in the previous chapter. And the argument which applied to $\{\wedge\}$ and $\{\vee\}$ works as well for $\{\rightarrow\}$ and $\{\leftrightarrow\}$. Recall that any formal sentence built from a wedge or vel (along with sentence letters and parentheses) will be **true in the first valuation**. But that holds as well for sentences built from an arrow or bicon.

●	▲	$(\bullet \rightarrow \blacktriangle)$	$(\bullet \leftrightarrow \blacktriangle)$
1	1	1	1
1	0	0	0
0	1	1	0
0	0	1	1

So no $\{\rightarrow\}$ or $\{\leftrightarrow\}$ sentence will match the truth table for a negation, which is false in the first valuation. Indeed, none of the following formal languages has a sentence matching a negation truth table; so **all are expressively inadequate**.

$\{\wedge, \vee, \rightarrow\}$	$\{\wedge, \vee\}$	$\{\wedge\}$
$\{\wedge, \vee, \leftrightarrow\}$	$\{\wedge, \rightarrow\}$	$\{\vee\}$
$\{\wedge, \rightarrow, \leftrightarrow\}$	$\{\vee, \rightarrow\}$	$\{\rightarrow\}$
$\{\vee, \rightarrow, \leftrightarrow\}$	$\{\wedge, \leftrightarrow\}$	$\{\leftrightarrow\}$
	$\{\vee, \leftrightarrow\}$	
	$\{\rightarrow, \leftrightarrow\}$	

The only remaining sub-language is $\{\sim, \leftrightarrow\}$. But we observe a remarkable feature of tildes and bicons in combination: if a biconditional or negation of one has more than one tilde, it is logically equivalent to either a biconditional with no tildes, or a biconditional with a single tilde.

For instance, “ $\sim(P \leftrightarrow Q)$,” “ $(\sim P \leftrightarrow Q)$,” and “ $(P \leftrightarrow \sim Q)$ ” all take the same truth table. So the result of adding further tildes to any of these sentences will yield a sentence that has the same truth table as either “ $(P \leftrightarrow Q)$ ” or “ $(P \leftrightarrow \sim Q)$ ”.

P	Q	$\sim P$	$\sim Q$	$(P \leftrightarrow Q)$	$\sim(P \leftrightarrow Q)$	$(\sim P \leftrightarrow Q)$	$(P \leftrightarrow \sim Q)$
1	1	0	0	1	0	0	0
1	0	0	1	0	1	1	1
0	1	1	0	0	1	1	1
0	0	1	1	1	0	0	0

For $\{\sim, \leftrightarrow\}$ sentences built from “P” and/or “Q,” the only further tables picked out are the contradiction truth table (taken by, e.g., “ $(P \leftrightarrow \sim P)$ ”), and the tautology truth table (taken by, e.g., “ $(P \leftrightarrow P)$ ”). So all “P” and “Q” sentences in the $\{\sim, \leftrightarrow\}$ language take one of these four truth tables, or the truth table for “P” or “Q” or their negations – eight truth tables in all.

P	Q	$\sim P$	$(P \leftrightarrow Q)$	$(P \leftrightarrow \sim Q)$	$(P \leftrightarrow P)$	$(P \leftrightarrow \sim P)$
1	1	0	1	0	1	0
1	0	0	0	1	1	0
0	1	1	0	1	1	0
0	0	1	1	0	1	0

Note that in this language, every sentence is true in an **even number of valuations**. That feature holds for all sentence letters, and negations and biconditionals of sentence letters; and will continue to hold for any larger biconditional or negation in this language. So that feature holds for all $\{\sim, \leftrightarrow\}$ sentences.

But a number of familiar truth tables don't have an even number of true valuations – e.g., the truth tables for “ $(P \wedge Q)$,” “ $(P \vee Q)$,” and “ $(P \rightarrow Q)$ ”. Offering no sentences which takes such a truth table, **the $\{\sim, \leftrightarrow\}$ language is expressively inadequate**. (Indeed, $\{\sim, \leftrightarrow\}$ is the **only** 2-connective sub-language of $\{\sim, \wedge, \vee, \rightarrow, \leftrightarrow\}$ which contains the tilde, but is still expressively inadequate.)

Summary

Expressive Power:
The Chapter Three Language and Its Sub-Languages:

