2.19. Features of Validity: Limit-Case Arguments

We've seen numerous arguments illustrating the nice parallel between validity, on the one hand, and intuitive "following from" or "entailing" on the other. But occasionally this parallel is strained – yielding arguments which do technically qualify as valid, but where we're uncomfortable saying that the conclusion follows from the premises.

And for all their variety, each of these peculiar cases stems from the nature of a **validity counterexample** for an argument: a valuation where (i) the **premises** of the argument **are all true**, but (ii) the **conclusion is false**. In what follows we survey several different and very unintuitive ways in which an argument can evade one or the other of these two conditions – thereby qualifying as a valid argument.

1. Unintuitive Arguments: Two Kinds. We first encountered tautologies and contradictions in our survey of formal translation. And even then, in advance of formal semantics, these two sorts of sentences stood out for sounding odd, and looking quite unlike the normal sentences we make in ordinary language. With that in mind, it's not especially surprising to find that arguments involving tautologies or contradictions yield results that can seem unnatural.

Note first that tautologies, as logically true, and contradictions, as logically false, are outliers even semantically. For in all other sentences we find that whether or not they're true in the actual world isn't simply a question of their **logical form**, but also of their **subject matter**. The sentence "It's raining and it's cold" has a certain logical form (it's a conjunction). But knowing whether it's actually true is a combination of that logical form and whether its atomic parts (the subject matter sentences "It's raining," "It's cold") are true in the actual situation: the conjunction is actually true if (and only if) both its parts are actually true, and those parts are actually rue if what they say (their meaning) matches the actual situation. (A conjunction built from different subject matter sentences – say "Neko's a strong swimmer and Jack plays the guitar" – will, because of that different subject matter, not be true in the same situations as "It's cold and it's raining".)

Since actual truth isn't, for such sentences, a matter of logical form alone, the **subject matter / logical form** distinction in that case parallels the two factors in having a convincing argument: **truth of the premises** and **validity of the**

argument. For logical form can here tell us whether the argument is valid, but not whether the premises are actually true.

We know already that tautologies and contradictions are the exception to the first point: with these limit-case sentences the **logical form alone** tells us whether that sentence is actually true or actually false. Since a tautology is true in every possible situation, it's bound to be true in the actual situation; and a contradiction is bound to be actually false for the same reason. But for that reason, in arguments involving tautologies and contradictions as premises or conclusion we can determine not only the validity of the argument, but the actual truth of the premises. Yet for all that we will find such arguments of dubious merit as concerns making their case convincingly.

Note first that **a tautology follows validly from any premises**. So the tautology "(~P ∨ P)" follows validly from, e.g., "X".

1. X

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∴ (P ∨ ~P)

	(1)		**∴**
P	**X**	~P	**(P ∨ ~P)**
1	**1**	0	**1**
1	0	0	1
0	**1**	1	**1**
0	0	1	1

Since a tautology is true in every valuation, an argument with a tautology as its conclusion is one whose conclusion is never false in any valuation. And that means no valuation can meet the second requirement for being a counterexample: making the conclusion false. **An argument with a tautology as conclusion is immune from any validity counterexample** – regardless of when its premise(s) are true, and when they're not, or what the subject matter of the argument is.

The mirror image of that case comes in an argument with a contradiction as a premise – or more generally, with inconsistent premises. For it turns out that **any conclusion follows validly from inconsistent premises**.

In the simplest case: an argument with a contradiction as premise is bound to be valid. So the argument "(P ∧ ~P) ∴ X" is valid.

1. (P ∧ ~P)

∴ X

			(1)	**∴**
P	X	~P	**(P ∧ ~P)**	**X**
1	1	0	0	1
1	0	0	0	0
0	1	1	0	1
0	0	1	0	0

There being no valuations which make the premises true, there are certainly none making (i) the premises true and (ii) the conclusion false. So **no counterexample is possible**.

Indeed, since counterexamples are blocked by the premise alone, it doesn't matter which sentence acts as conclusion. Hence **any and every sentence follows validly from a contradiction**. That makes a contradiction especially poisonous, in terms of entailment: assuming we do not believe that every sentence is true – and so don't wish to be committed to every sentence – logical insist that we must resist believing a contradiction.[1]

More generally: **any inconsistent set of premises will** likewise **validly entail any and every sentence**.

1. P
2. ~P

∴ X

(1)		**(2)**	**∴**
P	X	~P	**X**
1	1	0	**1**
1	0	0	0
0	1	1	**1**
0	0	1	0

The set of sentences {P, ~P} is unsatisfiable (inconsistent), since no valuation makes both sentences true. And once again, this means **no validity counterexample is possible**.

[1] Or at least: classical logic insists on this. There are varieties of "paraconsistent" logics in which contradictions are not so explosive concerning entailments.

2. Tautologies, Contradictions, and Validity. In considering what we are to make of these two mutant sorts of valid arguments, it's worthwhile combining our survey of argument validity with the earlier discussion of tautologies and contradictions. For this yields a classification of argument types with respect to the sort of premise or conclusion the argument has, and whether the argument is guaranteed to be valid or invalid as a result.

To this end we first introduce a simplified presentation of the premises of an argument: whenever an argument has more than one premise, we'll **conjoin all the premises** into one large conjunction. So the following familiar argument on the left has its 'conjunction counterpart' on the right.

1. $(P \vee Q)$ 2. $\sim P$	$((P \vee Q) \wedge \sim P)$
$\therefore Q$	$\therefore Q$

This is a harmless simplification semantically, since the concepts of both validity and valid counterexample involve the premises of an argument only concerning cases where **all** those **premises are true**. Since the conjunction of the premises will be true in just those situations where all the individual premises are true, conjoining the premises together will have no effect on whether the argument qualifies as valid or invalid.

With that simplification made, note first that every argument falls into one of three families, as concerns its (possibly conjoined) premise.

C: The premise is a contradiction.
N: The premise is neither a contradiction nor a tautology.
T: The premise is a tautology.

And an argument likewise falls into one of three families concerning its conclusion.

C: The conclusion is a contradiction.
N: The conclusion is neither a contradiction nor a tautology.
T: The conclusion is a tautology.

That means every argument can be classified according to its premise and conclusion. So, in "**Premise / Conclusion**" format, a **C/C** argument[2] is one where both premise and conclusion are contradictions, a **C/N** argument is one with contradiction as premise and conclusion neither tautology nor contradiction, and so on – yielding nine classes of arguments in total.

C/C	**C/N**	**C/T**
N/C	**N/N**	**N/T**
T/C	**T/N**	**T/T**

And for each of these classes of arguments except one, we find every argument in the class either **guaranteed to be valid** or **guarantee to be invalid**. That is: in such cases simply knowing what class the argument is in tells us whether it's valid.

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For example, **every C/ argument** (the top row of the list) **is guaranteed to be valid**, regardless of the status of its conclusion. For an argument is only invalid if there's a validity counterexample – a valuation where the argument's (possibly conjoined) premise is true but its conclusion is false. But with a PC argument there's no valuation where the premise is true – and so no valuation where that premises is true and the conclusion is false. A PC argument is immune to validity counterexamples.

Likewise **every /T argument** (the right column of the list) **is guaranteed to be valid**.

Every T/C argument is bound to be invalid. For such an argument every valuation is a validity counterexample.

Every N/C argument is bound to be invalid. For such an argument every valuation where the **premise is true** will be a validity counterexample.

For similar reasons **every T/N argument is bound to be invalid**.

That leaves only the central class: the **N/N** arguments. This is the only class of the nine that contains **some valid** arguments **and some invalid** arguments. N/N is the

[2] Pronounced "C-slash-C".

only class where membership in the class doesn't tell us in advance whether the argument is valid or invalid.

Of course the arguments the arguments that we make in everyday life fall practically entirely in this class – with premise(s) and conclusion of a factual nature, hence capable of being true but also capable of being false.[3] The eight surrounding classes are, by comparison, of merely theoretical interest – featuring as they do argument which are 'mutant' outliers by the standards of everyday discourse.

[In conclusion can already give one response to these odd cases: since tautology and contradiction cases already use sentences out of the ordinary, we can take a 'don't care' approach to formal logic's verdict in these cases. What remains are the other two cases: circularity and weakening. We return to these cases in pragmatics.]

3. Unintuitively Valid Arguments: Two More Kinds.

First, **any sentence follows validly from itself**. The following argument, for instance, is perfectly valid.

$$\frac{1.\ P}{\therefore\ P}$$

And we needn't bother with truth tables to see that this is so: in any possible situation (valuation) where "P" is true, "P" is true. It's thus impossible to have a validity counterexample for this argument – for any valuation clearing the first hurdle (making the premises true) fails on the second (making the conclusion false).

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[3] Keep in mind that a multi-premise argument is presented here as having a conjunction as premise, where each of the original premises is one (immediate) part of that conjunction. So even if one or more of the original premises are tautologies, so long **as least one** of the original premises **isn't** a tautology the whole conjoined premise won't be a tautology either. For (as noted in 2.17.1 Problem X) a conjunction is a tautology only if **all** its (immediate) parts are tautologies. So a **T/** argument is one in which **all** the original premises were tautologies.
By contrast: if **even one** of the original premises was a contradiction, the whole conjoined premise will likewise be a contradiction. So a **C/** argument is one where at least one of the original premises was a contradiction, or in which the original premise were mutually inconsistent – for example, "P . ~P ∴ X"

Of course this argument will never *convince* anyone of P, for reasons of pragmatics.[4] Briefly: anyone in need of convincing doesn’t already believe P. But not believing P, they will not judge the argument to pass the true premises requirement – and hence will not find the argument convincing.

***Second*, adding premises to a valid argument always yields a valid argument**. The following familiar argument, for instance, is at this point notoriously valid.

1. $(P \vee Q)$
2. $\sim P$

$\therefore Q$

And adding any premise whatsoever – however irrelevant – yields a larger valid argument.

1. $(P \vee Q)$
2. $\sim P$
3. X

$\therefore Q$

Truth tables bear this out: in the one valuation making all three premises true, the conclusion is true.

		(3)	(1)	(2)	$\therefore$
P	**Q**	**X**	**(P ∨ Q)**	**~P**	**Q**
1	1	1	1	0	1
1	1	0	1	0	1
1	0	1	0	0	0
1	0	0	0	0	0
0	1	**1**	**1**	**1**	**1**
0	1	0	**1**	**1**	1
0	0	1	0	1	0
0	0	0	0	1	0

[4] In Chapter 7, section X.

Note that the first two premises, "$(P \vee Q)$" and "$\sim P$," are true together in two valuations: the fifth (emphasized here) and the sixth. If adding an additional premise has any effect, it can only be (as in this case) to *reduce* the number of valuations making all the premises true. (Adding the third premise, "X," weeds out Valuation 6 as one making *all* the premises true.) By making it that much harder to have *all the premises true*, (the first requirement for a validity counterexample), we can only *lower* the chances of having of a validity counterexample. So if there were no counterexamples to begin with (because the original argument was valid), adding further premises cannot introduce a counterexample. The more premises we heap on, the more immune the argument becomes to counterexamples.