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GROUPS THAT ARE LOCALLY EMBEDDABLE IN THE CLASS OF FINITE GROUPS

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Introduction

In this paper we discuss various definitions and some properties of a certain class of groups, namely, the class of groups that are locally embeddable in the class of finite groups (the LEF-groups, for short). This class could have appeared as a specification of a certain quite general definition in model theory [1], but apparently it has not yet been considered either in group theory or in its applications. The first author came to it being involved in approximation theory of dynamical systems and operator algebras (see [2, 3, 4]); for the second author this class of groups arose from the study of approximation of operators on function spaces. A similar class of LEF-algebras (those locally embeddable in the class of finite-dimensional algebras) is probably even more interesting.

The problem of approximating infinite groups by finite ones has been investigated before in the framework of purely algebraic constructions (see, e.g., the classical papers [14, 15]). The notions introduced in what follows are more general; the classical algebraic notions of residual finiteness and inductive limit are proper special cases of our approximation. Roughly speaking, we use "near-homomorphisms" instead of homomorphisms.

The class of LEF-groups looks particularly important in ergodic theory, dynamical systems, and operator theory, where some interesting facts about the nature of approximation can be formulated in terms of this class. For example, for countable groups the notion of free approximation introduced in [2] and investigated in [8, 9, 10] is equivalent in some sense to the property of local embeddability in the class of finite groups (see §3 of this paper). It is remarkable that the class of LEF-groups is in general position relative to the class of amenable groups, and the class of finitely representable LEF-groups coincides with the class of finitely representable residually finite (RF) groups [9, 10]. In general, there exist solvable (hence, amenable) groups that are not LEF-groups; also, there are finitely generated LEF-groups that are not residually finite.

In this paper we present only some initial results on the LEF-groups. In §1 the basic definition of the class of LEF-groups is given, together with various statements equivalent to it. A relationship between the definition of approximation by finite groups introduced here and the convergence in the topological space of finitely generated groups (see [11, 12]) is investigated. The class of algebras that are locally embeddable in the class of finite-dimensional algebras is introduced, and a generalization of the definition of approximation by finite groups to the case of topological groups is formulated.

In §2 we collect certain facts pertaining to the structure of the class of LEF-groups. In particular, it is shown that all locally residually finite groups are LEF-groups, and

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all finitely representable LEF-groups are RF-groups. It turns out that the class of LEF-groups is not closed under extensions, nor even under semidirect products. But under the requirement that the action of an LEF-group A on an LEF-group B be approximable in some natural sense (see Definition 2.3([1])), the corresponding semidirect product $A \rightthreetimes B$ is also an LEF-group. This fact allows us to construct an example of an LEF-group with two generators that is not residually finite.

In §3 we study the approximability of the action of LEF-groups on arbitrary sets and also the quasiinvariant actions of such groups in spaces with probability measure. It is established that in both cases the class of LEF-groups is characterized by the existence of a freely approximable action. We formulate some open questions and conjectures related to the LEF-groups and algebras.

In our opinion, the general concept of convergence for the group operations (or in algebras, for semigroup operations) is of particular interest. For instance, approximation in the theory of Lie algebras can be treated precisely in this way, representing the infinite-dimensional Lie algebras with continuous systems of roots as the limits of classical Cartan algebras (see [13]).

§1. The basic definition and various versions of it

1°. Definition. We say that a group G is *locally embeddable in the class of finite groups* (G is an LEF-*group*) if for any finite set $H \subseteq G$ there exists a finite set K with $H \subseteq K \subseteq G$ and a binary operation $\odot\colon K^2 \longrightarrow K$ such that $(K, \odot)$ is a group satisfying the condition

$$\forall h_1, h_2 \in H\ (h_1 \cdot h_2 \in H \to h_1 \cdot h_2 = h_1 \odot h_2).$$

Here and in what follows, the group operation in G is denoted by dot. Also, we use the following abbreviations:

$\forall^f H \rightleftharpoons$ for every finite set H;

$\exists^f(K, \odot) \rightleftharpoons$ there exists a finite set K and a binary operation $\odot\colon K^2 \longrightarrow K$ turning K into a group.

Lemma. *A group G is an* LEF*-group if and only if the following condition is fulfilled: for any $k \in \mathbb{N}$,*

$$\forall^f H \subset G\ \exists^f(K, \odot)\ \left[H^k \subseteq K \wedge \forall h_1, \dots, h_k \in H\ (h_1 \cdot \dots \cdot h_k = h_1 \odot \dots \odot h_k)\right],$$

where $H^k = \{\, h_1 \cdot \dots \cdot h_k \mid h_i \in H \,\}$.

If $k = 2$, the condition of the lemma means that the Cayley table of G restricted to an arbitrary finite subset H of G, can be completed to the Cayley table of some finite group. It should be mentioned that the fragments of Cayley tables, i.e., the partitions of the set of pairs $(h_1, h_2) \in H \times H$ (where H is a finite subset of a group G) into the classes of pairs with the same product were considered in combinatorial group theory; see, e.g., the book [7], where the "doubling" problem was investigated.

It is easily seen that for countable groups the definition of an LEF-group can be reformulated as follows.

Proposition. *A countable group G is an* LEF*-group if and only if there exists an increasing sequence of finite sets $K_n \subseteq K_{n+1}$ such that $\bigcup_n K_n = G$ and on each K_n there is a binary operation $\odot_n$ turning $(K_n, \odot_n)$ into a group and satisfying*

$$\forall a, b \in G\ \exists N(a, b)\ \forall n > N(a, b)\ (a \odot_n b = a \cdot b).$$

([1])In the system of references adopted here, Definition 2.3 refers to the definition in Subection 3 of §2.

Sometimes, the latter condition is written in the form

$$a \cdot b = \lim_{n\to\infty} a \odot_n b.$$

We note that, in general, the group $(K_n, \odot_n)$ in this proposition is neither a subgroup of G nor a subgroup of $(K_{n+1}, \odot_{n+1})$.

The class of LEF-groups defined above may be called the class of *inductive* LEF-groups. It is possible to formulate also a *projective* analog of this definition. It turns out that the class of groups obtained in this way is the same as the previous one.

Theorem-definition. *For countable groups G, the following conditions are equivalent.*

1. *G is an* LEF*-group.*
2. *There exists a countable sequence of finite groups K_n and a system of maps (not necessarily homomorphisms) $\pi_n : G \longrightarrow K_n$ such that*
   a) $\forall x, y \in G\ (x \neq y \to \exists N\ \forall n > N\ \pi_n(x) \neq \pi_n(y))$;
   b) $\forall x, y \in G\ \exists N\ \forall n > N\ \pi_n(x \cdot y) = \pi_n(x) \cdot \pi_n(y)$.

Proof. 1. If G is an LEF-group, then we consider the sequence of finite groups K_n described in the preceding proposition and define $\pi_n : G \longrightarrow K_n$ by putting $\pi_n(x) = x$ for $x \in K_n$ and taking $\pi_n(x)$ arbitrarily otherwise. Obviously, condition 2 is fulfilled.

2. Conversely, let G satisfy condition 2. For an arbitrary finite set $H \subseteq G$, we choose n such that

i) $\forall x, y \in H\ (x \neq y \to \pi_n(x) \neq \pi_n(y))$,
ii) $\forall x, y \in H\ \pi_n(x \cdot y) = \pi_n(x) \cdot \pi_n(y)$.

Now we fix any map $j : K_n \longrightarrow G$ which is right inverse to π_n and satisfies $H \subset j(K_n)$. We put $K = j(K_n)$ and denote by $\odot$ the multiplication on K induced by j from K_n. Clearly, the condition occurring in the definition of an LEF-group is fulfilled. □

Remarks. 1. Proposition 1.1 and the theorem-definition remain valid for arbitrary (not necessarily countable) LEF-groups if we replace the sequences of finite groups by nets.

2. In the above theorem, if all π_n are homomorphisms, then we recover the classical notion of a residually finite group [14]. Thus, all RF-groups are LEF-groups (see Corollary 1 to Theorem 2.1 below).

3. Definition 1.1 readily shows that the class of LEF-groups (as well as the class of RF-groups) is hereditary, i.e., any subgroup of an LEF-group is an LEF-group.

2°. Local embeddability of algebras into finite-dimensional algebras can be defined by analogy with what was said in Subsection 1°. As before, another definition, similar to the theorem-definition of Subsection 1°, is possible here.

Definition. An algebra A is said to be *locally embeddable in the class of finite-dimensional algebras* (A is an LEF*-algebra*) if there exists a sequence of finite-dimensional spaces $A_n \subseteq A_{n+1}$ with $\bigcup_n A_n = A$ and bilinear operations $\circ_n : A_n^2 \longrightarrow A_n$ (thus, $(A_n, \circ_n)$ is an algebra) such that

$$\forall a, b \in A\ \exists N \forall n > N\ a \cdot b = a \circ_n b.$$

Theorem-definition. *The following two conditions are equivalent.*

1. *A is an* LEF*-algebra.*
2. *There exists a sequence of finite-dimensional algebras $(A_n, \circ_n)$ and a sequence of linear operators (not necessarily homomorphisms of algebras) $\varphi_n : A \longrightarrow A_n$ such that*
   a) $\forall a, b \in A\ (a \neq b \to \exists N\ \forall n > N\ \varphi_n(a) \neq \varphi_n(b))$,
   b) $\forall a, b \in A\ \exists N\ \forall n > N\ \varphi_n(a \cdot b) = \varphi_n(a) \circ_n \varphi_n(b)$.

Proof. The proof is similar to that of the theorem-definition in Subsection 1°, with the only difference that when constructing right inverse maps we must make them linear operators. This can always be done by using the direct complements of the subspaces A_n under consideration and the kernels of the homomorphisms φ_n. □

Proposition. *The group algebra of an* LEF*-group G is locally embeddable in the class of finite-dimensional algebras.*

Proof. For the finite-dimensional algebras A_n we can take the group algebras of the finite groups K_n of Proposition 1.1. □

The question as to whether the converse is true, i.e., whether the LEF property of a group G follows from the LEF property of its group algebra, remains open.

3°. The assumptions of Proposition 1.1 (or of Definition 1.2) can be slightly weakened. Clearly, instead of $K_n \subseteq G$ ($A_n \subseteq A$) we may require the existence of injective maps $j_n\colon K_n \longrightarrow G$ (injective linear operators $\psi_n\colon A_n \longrightarrow A$) that satisfy obvious modifications of the assumptions of Proposition 1.1 (respectively, Definition 1.2). It turns out that the condition of injectivity can also be lifted.

Proposition. i) *A countable group G is an* LEF*-group if and only if there exists a sequence of finite groups $\langle K_n, \odot_n\rangle$ together with maps $j_n\colon K_n \longrightarrow G$ such that*

1. $G = \varinjlim j_n(K_n)$ (*as before, here we mean the inductive limit of sets, not of groups*);
2. $\forall a, b \in G\ \exists N\ \forall n > N\ \forall \alpha, \beta \in K_n$

$$(j_n(\alpha) = a \wedge j_n(\beta) = b \rightarrow j_n(\alpha \odot_n \beta^{\pm 1}) = a \cdot b^{\pm 1});$$

3. $j_n(e_n) = e$, *where e_n (e) is the identity element of the group G_n (G).*

ii) *An algebra A is locally embeddable in the class of finite-dimensional algebras if and only if there exists a sequence of finite-dimensional algebras A_n and linear operators $\varphi_n\colon A_n \longrightarrow A$ (the φ_n are not necessarily homomorphisms of algebras) such that*

1. $A = \varinjlim \varphi_n(A_n)$ (*the inductive limit of linear spaces*);
2. $\forall a, b \in A\ \exists N\ \forall n > N\ \forall \alpha, \beta \in A_n$

$$(\varphi_n(\alpha) = a \wedge \varphi_n(\beta) = b \rightarrow \varphi_n(\alpha \circ_n \beta) = a \circ b),$$

where $\circ_n$ ($\circ$) denotes multiplication in the algebra A_n (A).

Proof. We restrict ourselves to statement i). Statement ii) can be proved similarly.

Obviously, every LEF-group satisfies the conditions listed in the proposition. To prove the converse, we fix an arbitrary finite set $H \subseteq G$. Conditions (1) and (2) imply the existence of n such that $j_n(K_n) \supseteq H$ and

$$\forall a, b \in H\ \forall \alpha, \beta \in K_n\ (j_n(\alpha) = a \wedge j_n(\beta) = b \rightarrow j_n(\alpha \odot_n \beta^{\pm 1}) = a \cdot b^{\pm 1}).$$

There is no loss of generality in assuming that $e \in H$. The conditions of our proposition imply that $\mathcal{M} = j_n^{-1}(\{e\})$ is a subgroup of K_n and that $H \subseteq j_n(\mathcal{N})$, where $\mathcal{N}$ is the normalizer of $\mathcal{M}$ in K_n. Let $K = \mathcal{N}/\mathcal{M}$, and let $\odot$ denote multiplication in K. It is easy to check that for every $\xi \in K$ we have $\#(j_n(\xi) \cap H) \leq 1$. We construct a map $j\colon K \longrightarrow G$ in the following way. Let $B = \{\xi \in K \mid \#(j_n(\xi) \cap H) = 1\}$. For any $\xi \in B$ we put $j(\xi) = h \in j(\xi) \cap H$. Since for any $\alpha, \beta \in \mathcal{N}$, $j_n(\alpha) = j_n(\beta) \in H$ implies $\alpha \odot_n \beta^{-1} \in \mathcal{M}$, the map j is injective on B. Since G can be assumed infinite (for the finite groups the statement is trivial), j can be extended to an injective map from K to G. Thus, we have constructed an *injective* map $j\colon K \longrightarrow G$ such that $H \subseteq j(K)$ and $j(\alpha \odot \beta) = j(\alpha) \cdot j(\beta)$ if $\alpha, \beta \in j^{-1}(H)$. Now it is obvious that the conditions of Lemma 1.1 are satisfied. □

4°. In [12], a certain natural topology in the space of groups with a fixed finite system of generators was investigated in detail; before, this topology had been used in [11].

Along the lines of [12], we describe the topological space $\mathcal{G}$ of groups with a system of generators $\langle a_1, \ldots, a_n \rangle$. Any group $G_0 \in \mathcal{G}$ is determined by a normal subgroup $\mathcal{N}(G_0)$ of the free group $\mathcal{F}(a_1, \ldots, a_n)$. The base of neighborhoods of G_0 consists of the sets $V_k(G_0) \subseteq \mathcal{G}$, $k \in \mathbb{N}$, given by

$$V_k(G_0) = \{ G \in \mathcal{G} \mid \mathcal{N}(G) \cap B_k = \mathcal{N}(G_0) \cap B_k \},$$

where B_k is the set of all words in $\mathcal{F}(a_1, \ldots, a_n)$ of length at most k.

It is easy to check (see [12]) that the space $\mathcal{G}$ is metrizable and compact.

Proposition. *A group G with generators $\langle a_1, \ldots, a_n \rangle$ is an* LEF*-group if and only if there exists a sequence of finite groups converging to G in the topological space $\mathcal{G}$.*

Proof. Let G be an LEF-group and $\pi \colon \mathcal{F}(a_1, \ldots, a_n) \longrightarrow G$ the natural homomorphism with $\operatorname{Ker} \pi = \mathcal{N}(G)$. Consider the finite set $H_k = \{ \pi(\alpha) \mid \alpha \in B_k \}$. Starting with H_k, we construct a finite group $\langle L_k, \odot \rangle$ that satisfies the conditions of Lemma 1.1. Obviously, we may assume that L_k is also generated by $\langle a_1, \ldots, a_n \rangle$. If $s \leq k$, then by Lemma 1.1 we have $a_{i_1}^{\delta_1} \cdot a_{i_2}^{\delta_2} \cdot \cdots \cdot a_{i_s}^{\delta_s} = a_{i_1}^{\delta_1} \odot a_{i_2}^{\delta_2} \odot \cdots \odot a_{i_s}^{\delta_s}$ with $\delta_i = \pm 1$. Hence, both expressions are or are not equal to the identity element simultaneously, i.e., $\mathcal{N}(G) \cap B_k = \mathcal{N}(L_k) \cap B_k$, which means that $L_k \to G$ as $k \to \infty$.

Conversely, suppose that a sequence of finite groups G_k satisfies $G_k \to G$ as $k \to \infty$. For an arbitrary finite set $H = \{h_1, \ldots, h_t\} \subseteq G$, we represent each $h_i \in H$ in any way as a word $h_i(a_1, \ldots, a_n) \in \mathcal{F}(a_1, \ldots, a_n)$. Consider the set of all words of the form $h_i(a_1, \ldots, a_n) \cdot h_j^{-1}(a_1, \ldots, a_n)$ with $i \neq j$ and of the form $h_i(a_1, \ldots, a_n) \cdot h_j(a_1, \ldots, a_n) \cdot h_k^{-1}(a_1, \ldots, a_n)$ with $h_i, h_j, h_k \in H$ satisfying $h_i \cdot h_j = h_k$. Let s be the maximum of the lengths of all such words. Then there exists a finite group $G_k \in V_s(G)$. Hence, $\mathcal{N}(G_k) \cap B_s = \mathcal{N}(G) \cap B_s$. Denoting by $\nu \colon \mathcal{F}(a_1, \ldots, a_n) \longrightarrow G_k = \mathcal{F}(a_1, \ldots, a_n)/\mathcal{N}(G_k)$ the natural projection, we define a map $\varphi \colon H \longrightarrow G_k$ by putting $\varphi(h) = \nu(h(a_1, \ldots, a_n))$. The map φ is injective, because $h_i \neq h_j$ implies $h_i(a_1, \ldots, a_n) \cdot h_j^{-1}(a_1, \ldots, a_n) \notin \mathcal{N}(G)$, whence $h_i(a_1, \ldots, a_n) \cdot h_j^{-1}(a_1, \ldots, a_n) \notin \mathcal{N}(G_k)$. Similarly, if $h_i \cdot h_j = h_k$, then $\varphi(h_i) \cdot \varphi(h_j) = \varphi(h_k)$. Now, using the facts that φ is injective and G is infinite, it is easy to construct a group $\langle K, \odot \rangle$ that satisfies the conditions of Definition 1.1. □

For the approximations studied in this paper it is more natural to introduce the space $\mathcal{S}$ of group operations (or Cayley tables). We restrict ourselves to at most countable groups, assuming that their elements are numbered by nonnegative integers in such a way that the group identity element always has index 0. Moreover, we agree that if a group is countable, then its numbering is bijective, and if its cardinality is n, then its elements are numbered by $0, 1, \ldots, n-1$, and all elements with indices exceeding n are equal to the identity element. Then $\mathcal{S}$ consists of all maps $f \colon \mathbb{N}^2 \longrightarrow \mathbb{N}$ that, in accordance with the above, define a group structure on $\mathbb{N}$ with zero as the identity. If $H \subseteq \mathbb{N}$, we put

$$V_H(f) = \{ f' \in \mathcal{S} \mid f'|H^2 = f|H^2 \}.$$

Obviously, the family $\{ V_H(f) \mid f \in \mathcal{S}, H \subseteq \mathbb{N}, \#(H) < \infty \}$ is a base of some topology on $\mathcal{S}$, which, as in [10], will be called the Cayley topology.(2) It is easily seen that the LEF-groups (and only the LEF-groups) are the limits of sequences of finite groups in the topological space $\mathcal{S}$.

(2)In [9], the Cayley topology was defined similarly in the space of groups with a fixed finite system of generators.

We note that the above proposition does not imply the existence of any continuous embedding of the space $\mathcal{G}$ in the space $\mathcal{S}$. The reason is that the relation of group isomorphism does not have any good topological properties in either of the topologies (see [11]).

5°. Now we present a definition of approximation of topological groups by finite ones. This definition is equivalent to that introduced in [6]. Below it is assumed that the topological group G in question is locally compact and metrizable, and that the topology of G is determined by a left-invariant metric ρ.

Definition. A locally compact group G is said to be *approximable* (by finite groups) if for any $\varepsilon > 0$ and any compact set $H \subseteq G$ there exists a finite group $(K, \odot)$ and an injective map $j\colon K \longrightarrow G$ with the following properties.

1. $j(K) \cap H$ is an ε-net in H.
2. For any $k_1, k_2 \in j^{-1}(H)$ we have

$$\rho\left(j(k_1 \odot k_2^{\pm 1}), j(k_1) \cdot j(k_2)^{\pm 1}\right) < \varepsilon.$$

3. $j(e_K) = e_G$.

It is easily seen that for the discrete groups this definition is equivalent to that of LEF-group. For the compact groups, the formulation of this definition simplifies slightly, namely, a compact group G is approximable if and only if for any $\varepsilon > 0$ there exists a finite group $(K, \odot)$ and an injective map $j\colon K \longrightarrow G$ such that

1. $j(K)$ is an ε-net in G;
2. $\forall k_1, k_2 \in K\ \rho(j(k_1 \odot k_2^{\pm 1}), j(k_1) \cdot j(k_2)^{\pm 1}) < \varepsilon$;
3. $j(e_K) = e_G$.

In the case of a separable group G, this definition can be reformulated in a form similar to Proposition 1.1.

Proposition. *A separable locally compact group G is approximable if and only if there exists a sequence of finite groups $(K_n, \odot_n)$ and a sequence of maps $j_n\colon K_n \longrightarrow G$ with the following properties*:

1. $\forall \xi \in G\ \forall \varepsilon > 0\ \exists N\ \forall n > N\ \exists h \in K_n\ (\rho(j_n(h)), \xi) < \varepsilon)$;
2. *for any $\varepsilon > 0$ and any compact set $H \subseteq G$*,

$$\exists N\ \forall n > N\ \forall k_1, k_2 \in j_n^{-1}(H)\ \rho\left(j_n(k_1 \odot k_2^{\pm 1}), j_n(k_1) \cdot j_n(k_2)^{\pm 1}\right) < \varepsilon;$$

3. $j_n(e_{K_n}) = e_G$.

Remark. If a group G is nonmetrizable, in the above definition we must replace an arbitrary $\varepsilon > 0$ by an arbitrary neighborhood of the identity element of G.

The question as to what topological groups are approximable by finite groups is still open. In [5], in nonstandard analysis terms it was established that this is true for all locally compact Abelian groups. It is easy to check that the (topological) inductive and projective limits of finite groups are approximable. In [6] it was proved that any approximable locally compact Abelian group is unimodular. But this condition is not sufficient; indeed, later it will be shown that there exist discrete groups that are not LEF-groups. Corollary 5 to Theorem 2.1 below shows that there exist locally compact groups not approximable as topological groups that are approximable as discrete groups, i.e., are LEF-groups. The following theorem provides a sufficient condition ensuring that discrete approximability implies topological approximability.

Theorem. *If for any neighborhood U of the identity element of a locally compact group G there exists a discrete subgroup $T \subseteq G$ that is an* LEF*-group satisfying $U \cdot T = G$, then G is topologically approximable by finite groups.*

Proof. We need to check that the requirements of Definition 1.5 are satisfied. Let $U = \{\xi \in G \mid \rho(\xi, e) < \varepsilon\}$, let T satisfy the conditions of the theorem, and let K be any compact subset of G. There exists a finite set $F \subseteq T$ such that $U \cdot F \supseteq K$. Moreover, since T is discrete, $T \cap K = S$ is finite. By Lemma 1.1, there is a finite group $(K, \odot)$ such that $(S \cup F)^2 \subseteq K$ and $h_1 \cdot h_2^{\pm 1} = h_1 \odot h_2^{\pm 1}$ for any $h_1, h_2 \in S \cup F$. Taking as j the identical embedding, we see that the requirements of Definition 1.5 are satisfied. Indeed, the first of them follows from the definition of U and the left invariance of the metric ρ, and the second from the fact that $j^{-1}(H) = S$ in this case, and for both products involved we have exact equality (in S). □

6°. For topological algebras, the definitions of injective and projective approximability by finite-dimensional algebras are substantially different. In what follows it will be shown that the group algebras of approximable topological groups are projectively approximable by group algebras of finite groups.

For completeness, we start with the definition of inductively approximable algebras, which generalizes the well-known notion of AF-algebras (see [3, 4]).

Definition 1. A topological algebra A is said to be *inductively approximable* (by finite-dimensional algebras) if there exists an increasing sequence of finite-dimensional subspaces A_n of A, each equipped with a continuous bilinear operation $\circ_n : A_n^2 \longrightarrow A_n$, such that $(A_n, \circ_n)$ is an algebra satisfying the following conditions:

1. $\bigcup_{n \in \mathbb{N}} A_n$ is dense in A;
2. for any $a, b \in \bigcup_{n \in \mathbb{N}} A_n$ we have $a \cdot b = \lim_{n \to \infty} a \circ_n b$.

Now we give the definition of projective approximability for the case of Banach algebras.

Definition 2. We say that a Banach algebra A is *projectively approximable* by finite-dimensional algebras $(A_n, \circ_n)$, if there exist a dense subalgebra $B \subseteq A$ and linear operators $\varphi_n : B \longrightarrow A_n$ such that

1. $\forall a \in B \; \|a\| = \lim_{n \to \infty} \|\varphi_n(a)\|_n$;
2. $\forall a, b \in B \; \lim_{n \to \infty} \|\varphi_n(a \cdot b) - \varphi_n(a) \circ_n \varphi_n(b)\|_n = 0$.

Remark. If A is a $*$-algebra, then it is approximable by finite-dimensional $*$-algebras in the sense of each of the above definitions provided that we have the same convergence for involutions as for the product.

Proposition. *If a separable locally compact group G is approximable by finite groups K_n, then the group algebra $L_1(G)$ is projectively approximable by the group algebras $A(K_n)$.*

Proof. In [6], the following facts were established. If a locally compact separable group G is approximable by finite groups, then G is unimodular. Fix any Haar measure μ on G. If U is a relatively compact neighborhood of the identity element and $\Delta_n = \mu(U)/\#(j_n^{-1}(U))$, then for any μ-almost everywhere continuous bounded function f decreasing sufficiently fast at infinity (in particular, for the functions with compact support) we have

$$\int_G f d\mu = \lim_{n \to \infty} \Delta_n \cdot \sum_{g \in K_n} f(j_n(g)).$$

We introduce a norm on $A(K_n)$ by putting

$$\|a\|_n = \Delta_n \cdot \sum_{g \in K_n} |a(g)|, \quad a \in A(K_n).$$

If, for example, we take $C_0(G)$ as B and introduce the operators $\varphi_n \colon C_0(G) \longrightarrow A(K_n)$ that map any function $f \in C_0(G)$ to the table of the values of f at the points of the set $j(K_n)$, i.e.,

$$\forall k \in K_n \; \forall f \in C_0(G) \; \varphi_n(f) = f(j_n(k)),$$

then the requirements of Definition 2 will be satisfied. □

§2. The structure of the class of LEF-groups

1°. First of all, we prove a theorem that identifies the class of LEF-groups in traditional algebraic terms. This theorem shows that this class is close to the class of locally residually finite (LRF) groups. Recall that a group G is residually finite if and only if for any finite set $H \subseteq G$ there exists a homomorphism of G onto some finite group such that the restriction of this homomorphism to H is injective. A group is said to be LRF if every finitely generated subgroup of it is RF. We denote by $G(H)$ the subgroup of G generated by H.

Theorem. *A group G is an* LEF*-group if and only if for any finite set $H = \{h_1, \dots, h_t\} \subseteq G$ there exists a group F, a finite group L, and surjective homomorphisms $\psi \colon F \longrightarrow G(H)$ and $\varphi \colon F \longrightarrow L$ such that for some $f_1, \dots, f_t \in F$ satisfying $\psi(f_i) = h_i$, $i = 1, \dots, t$, the homomorphism φ is injective on $\{f_1, \dots, f_t\}$.*

Proof. First, let G be an LEF-group. Let F denote the group with the generators $h_1, \dots, h_t$ and the determining relations $h_i \cdot h_j = h_k$ whenever this is fulfilled in G (there are only finitely many such relations because H is a finite set). We denote by D the set of all such relations. Since all relations belonging to D are valid in $G(H)$, there exists a surjective homomorphism $\psi \colon F \longrightarrow G(H)$. Now, let $\langle K, \odot \rangle$ be a finite group constructed starting with H in accordance with Definition 1.1. We may assume that K is generated by H. Definition 1.1 implies that all relations belonging to D are valid in K, i.e., there exist a surjective homomorphism $\varphi \colon F \longrightarrow K$. The elements $h_1, \dots, h_t$ are pairwise distinct in each of the groups $G(H), F$, and K. Moreover, for $i \le t$ we have $\varphi(h_i) = \psi(h_i) = h_i$; this proves the "only if" part of our theorem (it suffices to put $K = L$ and $f_i = h_i$, $i = 1, \dots, t$).

Let us prove the "if" part. Without loss of generality we may assume that the group $G(H)$ is infinite. Let $L = \{s_1, \dots, s_t, s_{t+1}, \dots, s_n\}$, where $s_i = \varphi(f_i)$ for all $i \le t$. We show that $f_{t+1}, \dots, f_n$ can be chosen in F in such a way that $\varphi(f_i) = s_i$ for all $i \in [t+1, n]$ and any elements $h_1, \dots, h_t, \psi(f_{t+1}), \dots, \psi(f_n)$ are pairwise distinct in $G(H)$.

Suppose that

$$\forall f \in \varphi^{-1}(s_{t+1}) \; \exists i \le t \; \psi(f) = h_i. \tag{1}$$

We put $N = \operatorname{Ker} \psi$, $M = \operatorname{Ker} \varphi$. Then (1) means that $\varphi^{-1}(s_{t+1}) = aM \subseteq h_1 N \cup \dots \cup h_t N$, whence

$$M \subseteq a^{-1} h_1 N \cup \dots \cup a^{-1} h_t N. \tag{2}$$

Since M is of finite index (the group L is finite), so is N, which contradicts the fact that $G(H)$ is infinite. Thus, statement (1) fails, and there is an element $f_{t+1} \in F$ such that $\varphi(f_{t+1}) = s_{t+1}$ and the element $\psi(f_{t+1}) = h_{t+1}$ is equal to none of the h_i with $i \le t$. This argument can be continued by induction. Now, on the set $K = \{h_1, \dots, h_t, \psi(f_{t+1}), \dots, \psi(f_n)\}$, whose cardinality is equal to that of L, we introduce the group structure $\odot$ induced from L. The construction of K shows that $H \subseteq K$ and

$$\forall a, b \in K \; (a \cdot b \in K \rightarrow a \cdot b = a \odot b), \tag{3}$$

so that the requirements of Definition 1.1 are fulfilled. □

A finite group $\langle K, \odot \rangle$ satisfying condition (3) is called an *approximating group* for G. The above theorem readily implies the following statement.

Corollary 1. *Every* LRF*-group is an* LEF*-group. In particular, all free groups are* LEF.

Corollary 2. *The following classes of groups are contained in the class of* LEF*-groups:*
1. *Abelian groups;*
2. *locally finite groups;*
3. *nilpotent groups;*
4. *matrix groups;*
5. *metabelian (solvable of degree* 2*) groups.*

This corollary follows from the fact that all classes listed above are contained in the class of LRF-groups (information concerning the matrix groups can be found in [14]; for metabelian groups, see [18]).

Corollary 3. *A group* G *is an* LEF*-group if and only if the following condition is fulfilled (cf. Definition* 1.1*):*

$$\forall^f H \subset G\, \exists^f (K, \odot)\, \big[H \subseteq K \subseteq G \wedge \forall h_1, h_2 \in K\, (h_1 \cdot h_2 \in K \to h_1 \cdot h_2 = h_1 \odot h_2)\big].$$

Proof. This follows from the construction of the group $(K, \odot)$ in the proof of the theorem (see condition (3)). □

For countable groups, the above corollary offers a possibility to formulate a refinement of Proposition 1.1.

Corollary 4. *A countable group* G *is an* LEF*-group if and only if there exists a sequence of finite groups* $\langle L_n, \odot_n\rangle$ *with the following properties:*
1. $L_n \subseteq L_{n+1}$ *(inclusion of sets, but not of groups!);*
2. $\bigcup_{n\in\mathbb{N}} L_n = G$;
3. $\forall a, b \in L_n\ (a \cdot b \in L_n \to a \cdot b = a \odot b)$.

Corollary 5. *There exist locally compact* LEF*-groups that are not approximable as topological groups (i.e., not approximable in the sense of Definition* 1.5*).*

Proof. Let G be the semidirect product of the additive group $\mathbb{R}$ and the multiplicative group $\mathbb{R}_+$. Then G is an LEF-group by part 4 of Corollary 2. It is well known that G is not unimodular, so that it is not approximable as a topological group. □

Corollary 6. *If* G *is a simply connected nilpotent Lie group whose Lie algebra* $\mathfrak{g}$ *has a basis with rational structure constants, then* G *is approximable as a topological group (in the sense of Definition* 1.5*).*

Proof. It is well known (see, e.g., [19]), that under the assumptions of this corollary the group G possesses lattices, and each lattice can be constructed in the following way. Let $\mathfrak{g}_0$ be the Lie algebra over $\mathbb{Q}$ generated by a basis in $\mathfrak{g}$ with rational structure constants, and let L be a lattice of maximal rank in $\mathfrak{g}$ contained in $\mathfrak{g}_0$. Then the subgroup $H \subseteq G$ generated by $\exp L$ is a lattice in G.

Obviously, for any neighborhood of zero $V \subseteq \mathfrak{g}$ there exists a lattice L satisfying these conditions and such that $V + L = \mathfrak{g}$. Since for any simply connected nilpotent Lie group G the map $\exp\colon \mathfrak{g} \longrightarrow G$ is an analytic isomorphism of manifolds, we easily deduce that for any neighborhood $U \subseteq G$ of the identity the desired lattice $L \subseteq \mathfrak{g}$ can be chosen in such a way that $U \cdot \exp L = G$. Then also $U \cdot H = G$, where H is the subgroup generated by $\exp L$. Moreover, the group H is nilpotent as a subgroup of a nilpotent group, whence, by Corollary 2, it is an LEF-group. Now the corollary follows from Theorem 1.5. □

As an example we consider topological approximation of the nilpotent group $UT(n, \mathbb{R})$.

We recall that if K is a commutative associative ring with identity, then the group of all $(n \times n)$-matrices over K with zeros under the principal diagonal and units on the principal diagonal is denoted by $UT(n, K)$. We write the elements of this group in the form $x = (x_{ij})_{1<i<j\leq n}$. If $y = (y_{ij}) \in UT(n, K)$, then it is convenient to write the formula for the elements of the product $z = (z_{ij}) = xy$ in the form

$$z_{ij} = x_{ij} + y_{ij} + \sum_{k=1}^{j-1} x_{ii+k} y_{i+kj}.$$

So, let $G = UT(n, \mathbb{R})$. Since this group is separable, we can construct a sequence of finite groups G_N satisfying the conditions of Proposition 1.5. Let $G_N = UT(n, \mathbb{Z}_M)$, where $M = 2N^n + 1$, and $\mathbb{Z}_M$ is the ring of residues mod M; here it is convenient to represent the latter ring as the ring of least absolute value residues, i.e., $\mathbb{Z}_M = \{-N^n, \dots, N^n\}$.

We define an injective map $j_N : G_N \longrightarrow G$ as follows. If $k = (k_{ij}) \in G_N$, i.e., $k_{ij} \in \mathbb{Z}_M$, then $j_N(k) = x = (x_{ij})$, where

$$x_{ij} = \frac{k_{ij}}{N^{j-i}}.$$

Obviously, the third condition of Proposition 1.5 is fulfilled. In order to check the first two conditions, we start with the observation that we may use any (not necessarily left-invariant) metric that determines the topology of the group $UT(n, \mathbb{R})$. In particular, we shall use the metric $\rho(x, y) = \max |x_{ij} - y_{ij}| = |x - y|$.

We verify the first condition of Proposition 1.5. Let $\xi = (\xi_{ij}) \in UT(n, \mathbb{R})$, and let $\varepsilon > 0$. Putting $N_0 = [\max\{\xi_{ij}, \varepsilon^{-1}\}]$, for any $N > N_0$ we find an element $k = k_{ij} \in G_N$ such that

$$\frac{k_{ij}}{N^{j-i}} \leq \xi_{ij} < \frac{k_{ij}+1}{N^{j-i}}.$$

The choice of N readily shows that $|k_{ij}| < N^n$ (i.e., we really have $k \in G_N$), and that $\rho(\xi, j_N(k)) < \varepsilon$.

To verify the second condition of Proposition 1.5, we note that it suffices to consider the compact sets of the form $\Delta_a = \{\, k \in G \mid |k_{ij}| \leq a \,\}$, $a \in \mathbb{R}_+$. Let $N > a$, and let $k = (k_{ij}) \in G_N$. It is easily seen that $|k_{ij}| < N^n$ if $j_N(k) \in \Delta_a$. Since the products in $\mathbb{Z}$ and in $\mathbb{Z}_M$ of any two numbers satisfying the latter inequality coincide, the above formula for the product of unipotent matrices allows us to show that

$$\forall k, s \in G_N \; (j_N(k), j_N(s) \in \Delta_a \to j_N(k \cdot s) = j_N(k) \cdot j_N(s)). \qquad \square$$

2°. The following theorem (essentially contained in [8]) provides examples of groups not embeddable in the class of finite groups.

Theorem. *Any finitely representable* LEF*-group is residually finite.*

Proof. Let $\{h_1, \dots, h_t\}$ be a set of generators of a group G and D a finite set of its determining relations. Given a nonempty finite set $H \subseteq G$, we represent each element of H as a word in the alphabet $A = \{h_1^{\pm 1}, \dots, h_t^{\pm 1}\}$. Let s denote the maximum length of a word in the set $D \cup H$. Let $L = A \cup \{e\}$. Lemma 1.1 implies the existence of a finite group $\langle K, \odot \rangle$ such that $L^s \subseteq K \subseteq G$ and

$$\forall a_1, \dots, a_s \in L \; (a_1 \cdot \dots \cdot a_s = a_1 \odot \dots \odot a_s). \tag{4}$$

We may assume that K is generated by the elements $h_1, \dots, h_t$. From (4) it follows that $H \subseteq K$ and that all determining relations of G are valid in L, i.e., there exists a surjective homomorphism $\varphi : G \longrightarrow L$. By (4), this homomorphism maps all elements of the set H into themselves, so that they remain pairwise distinct; this proves that G is residually finite. $\square$

Corollary 1. *The finitely representable groups with undecidable word identity problem are not* LEF*-groups.*

Proof. The finitely representable RF-groups have decidable word identity problem by a well-known theorem of A. I. Mal'tsev (see [15]). □

Corollary 2. *There exist solvable (hence, amenable) groups that are not* LEF*-groups.*

Proof. An example of finitely representable solvable group (of degree 3) with undecidable word identity problem was constructed by Kharlampovich in [16]. □

Remark. On the other hand, Corollary 1 to Theorem 2.1 shows that there are nonamenable LEF-groups (for example, free groups). Thus, the class of amenable groups is in general position relative to the class of LEF-groups. Therefore, it seems of interest to distinguish amenable groups in the class of LEF-groups. In particular, it is interesting to investigate the relationship between amenability and the asymptotic behavior of the function

$$\psi(n) = \frac{\#\{\, \langle a, b\rangle \in L_n^2 \mid a \cdot b \in L_n \,\}}{(\# L_n)^2},$$

where the L_n are the finite groups occurring in Corollary 4 to Theorem 2.1. It is easy to show that if G is an Abelian group, then the sequence of L_n's can be chosen so that $\psi(n) \to c > 0$ as $n \to \infty$.

Corollary 3. *No finitely representable infinite simple group is an* LEF*-group (see* [17] *for the existence of such groups).*

Proof. It is clear that no infinite simple group is residually finite. □

Corollary 4. *No finitely representable non-Hopfian group is an* LEF*-group.*

The simplest example of such a group can be found in [16]. This is the group $G = \langle b, t; t^{-1}b^2t = b^3\rangle$.

3°. In order to investigate the question as to whether the property of local embeddability in the class of finite groups is stable under extensions, it is natural to define approximability of the action of an LEF-group Π on an LEF-group G.

Definition. Let G and Π be LEF-groups. An action $\varphi\colon \Pi \longrightarrow \mathrm{Aut}(G)$ is said to be *approximable* if $G \leftthreetimes_\varphi \Pi$ is an LEF-group, and *equivariantly approximable* (e.a.) if for any two finite sets $H \subseteq G$ and $S \subseteq \Pi$ there are finite groups $\langle K, \odot_K\rangle$ and $\langle L, \odot_L\rangle$ containing H and K, respectively, and satisfying the conditions of Definition 1.1, and an action of L on K by automorphisms $\psi\colon L \longrightarrow \mathrm{Aut}(K)$ such that

$$\forall \pi \in S\ \forall g \in H\ (\varphi(\pi)(g) \in H \to \varphi(\pi)(g) = \psi(\pi)(g)).$$

Proposition. *If G and Π are* LEF*-groups, and $\varphi\colon \Pi \longrightarrow \mathrm{Aut}(G)$ is an e.a. action, then the semidirect product $G \leftthreetimes_\varphi \Pi$ is an* LEF*-group.*

Proof. Let $F = \{\langle g_1, \pi_1\rangle, \dots, \langle g_n, \pi_n\rangle\} \subseteq G \leftthreetimes_\varphi \Pi$. We put $H_1 = \{g_1, \dots, g_n\}$, $S = \{\pi_1, \dots, \pi_n\}$. There is no loss of generality in assuming that $H_1 = H_1^{-1}$, $e \in H_1$. Denoting $H = H_1^2$, for H and S we construct the groups K and L described in the above definition. Let $P = K \leftthreetimes_\psi L$. Then $F \subseteq P \subseteq G \leftthreetimes_\varphi \Pi$, and the pair (F, P) satisfies Definition 1.1. □

The following example of an approximable action was considered earlier in [2, 8].

Let $\mathrm{Symm}(\mathbb{Z})$ be the group of finite permutations of $\mathbb{Z}$. Obviously, $\mathrm{Symm}(\mathbb{Z})$ is an LEF-group, because it is locally finite. The action of $\mathbb{Z}$ on itself by shifts generates the action of $\mathbb{Z}$ on $\mathrm{Symm}(\mathbb{Z})$ by automorphisms: for any $\pi \in \mathrm{Symm}(\mathbb{Z})$ and any $m, n \in \mathbb{Z}$,

$$\pi^{(m)}(n) = \pi(n+m) - m. \tag{5}$$

We show that this action is e.a. Indeed, let $H \subseteq \mathbb{Z}$, $S \subseteq \mathrm{Symm}(\mathbb{Z})$ be any two finite sets. We choose $m \in \mathbb{N}$ such that

1. $H \subseteq \{-n_0, -n_0+1, \dots, n_0-1, n_0\}$;
2. $\forall \pi \in S\ \forall m \in \mathbb{Z}\ (|m| > n_0 \to \pi(m) = m)$.

Let $K = \mathbb{Z}/(2n_0+1)\mathbb{Z} = \{-n_0, \dots, n_0\}$ and let $L = \mathrm{Sym}(K)$ be the group of permutations of the finite set K. Then $H \subseteq K$, $S \subseteq L$ by construction, and it is clear that K and L are approximating groups for $\mathbb{Z}$ and $\mathrm{Symm}(\mathbb{Z})$, respectively ($L$ is even a subgroup of $\mathrm{Symm}(\mathbb{Z})$). As above, the action of K on itself by shifts $\mod (2n_0+1)$ induces the action of K on L by automorphisms: for any $\pi \in L$ and any $m, n \in K$,

$$\pi^{[m]}(n) = \pi(n \oplus m) \ominus m,$$

where $\oplus$ and $\ominus$ denote addition and subtraction mod $(2n_0+1)$. It is easy to verify that

$$\forall m, n \in K\ \bigl(\pi^{(m)}(n) \in K \to \pi^{(m)}(n) = \pi^{[m]}(n)\bigr).$$

This proves that the action (5) of the group $\mathbb{Z}$ on $\mathrm{Symm}(\mathbb{Z})$ is equivariantly approximable.

Thus, the semidirect product $G = \mathrm{Symm}(\mathbb{Z}) \leftthreetimes \mathbb{Z}$ is an LEF-group. Since $\mathrm{Symm}(\mathbb{Z})$ contains the infinite simple subgroup of even permutations, the group G is not residually finite. On the other hand, it is easy to check that this group is generated by the two elements $\langle 1, \mathrm{id}\rangle$ and $\langle 0, (1,2)\rangle$, where id is the identical permutation and $(1,2)$ is a transposition. Thus, we have proved the following statement.

Theorem (see [9]). *There exist finitely generated* LEF*-groups that are not residually finite.*

Corollary. *The class of* LRF*-groups is a proper subclass of the class of* LEF*-groups.*

Another example of such a group is the direct 'wreath product $\mathbb{Z} \operatorname{wr} \mathrm{Aff}(\mathbb{Q})$. In [18] it was shown that this group with three generators is not residually finite. On the other hand, Theorem 2.4 (see below) implies that this is an LEF-group. Thus, we have an example of a solvable group of degree 3 that is an LEF-group but is not residually finite.

4°. The existence of solvable non-LEF-groups (see Corollary 2 to Theorem 2.2) shows that the class of LEF-groups is not closed under extension; moreover, there exists an extension of an LEF-group by an Abelian group that is not an LEF-group. It seems of interest to find out what conditions ensure that the LEF property be preserved under extension.

Theorem. i) *The Cartesian product of an arbitrary family of* LEF*-groups is an* LEF*-group.*

ii) *If A and B are* LEF*-groups, then their direct wreath product $A \operatorname{wr} B$ is an* LEF*-group.*

iii) *If A is an* LEF*-group and B is a locally finite group, then the Cartesian wreath product $A \operatorname{Wr} B$ is an* LEF*-group.*

Lemma. *Assume that for any finite subset H of a group G there exists a homomorphism injective on H of the subgroup $G(H)$ generated by H into some* LEF*-group. Then G is an* LEF*-group.*

Proof of the theorem. Obviously, the class of LEF-groups is closed under finite Cartesian products. Moreover, it is clear that for any finite subset H of an infinite Cartesian product $\prod_{\alpha\in A} G_\alpha$ of groups there exists a finite subset $A_0 \subseteq A$ such that the projection of $\prod_{\alpha\in A} G_\alpha$ onto $\prod_{\alpha\in A_0} G_\alpha$ is injective on H. Now, statement i) follows immediately from the lemma.

In what follows, e' and e are the identity elements of the groups A and B, respectively.

We remind the reader that the Cartesian wreath product $A \operatorname{Wr} B$ is the semidirect product $A^B \leftthreetimes_\varphi B$ relative to the action φ of the group B on A^B by shifts: for any $f \in A^B$, $(\varphi_b f)(x) = f(bx)$ for all $b, x \in B$.

It is easily seen that the subgroup $\operatorname{Fin}(A, B) \subset A^B$ defined by

$$\operatorname{Fin}(A,B) = \{\, f \in A^B \mid \#\{\, b \in B \mid f(b) \neq e' \,\} < \infty \,\}$$

(the direct sum) is invariant under the action φ. The semidirect product $\operatorname{Fin}(A,B) \leftthreetimes_\varphi B$ is the direct wreath product $A \operatorname{wr} B \subset A \operatorname{Wr} B$. We show that the action φ of the group B on $\operatorname{Fin}(A, B)$ is equivariantly approximable; then statement ii) will follow from Proposition 2.3.

Let $S \subseteq B$, $H \subseteq \operatorname{Fin}(A, B)$ be finite sets. For $f \in \operatorname{Fin}(A, B)$, we denote by $\operatorname{supp} f$ the set $\{\, b \in B \mid f(b) \neq e' \,\}$ and put $T = \bigcup_{f\in H} (\operatorname{supp} f \cup (\operatorname{supp} f)^{-1})$. Let

$$S_0 = S \cup S^{-1} \cup T \cup \{e\}, \qquad S_1 = S_0^3.$$

Starting with S_1, we construct a finite group L satisfying Definition 1.1 for B. We put $M = \bigcup_{f\in H} \operatorname{im} F$ and, starting with M, construct a finite group F satisfying Definition 1.1 for A. Let $K = F^L$. We may assume that $K \subseteq \operatorname{Fin}(A, B)$ (as a set), extending the elements of K to the points of $B \setminus L$ by the identity element e' of the group A. Then $H \subseteq K$ by construction. Let $\psi\colon L \longrightarrow \operatorname{Aut}(F^L)$ be the action of L on F^L by shifts, i.e., for $b \in L$ and $f \in K$, $\psi_b f(x) = f(b \odot x)$ for all $x \in L$, where $\odot$ denotes multiplication in L. If $x \notin L$, then $\psi_b f(x) = e'$ because $f \in K$, and the elements of K are extended to the points of $B \setminus L$ by the identity element.

Let $b \in S$, $f \in H$ be arbitrary; we show that $(\varphi_b f)(x) = (\psi_b f)(x)$ for any $x \in B$. If so, then the conditions of Definition 2.3 will be fulfilled (and even stronger conditions, because we do not require that $\varphi_b f \in H$). The following two cases are possible.

1) $b \cdot x \in T$. Then $x = b^{-1} \cdot b \cdot x \in S_0^2 \subseteq S_1$, i.e., $x \in S_1$. But $b \in S$, i.e., $b \in S_1$, and since $b \cdot x \in S_1$ and L satisfies Definition 1.1 precisely for S_1, we have $b \cdot x = b \odot x$, i.e., $(\varphi_b f)(x) = (\psi_b f)(x)$.

2) $b \cdot x \notin T$. Then $f^b(x) = f(b \cdot x) = e'$ because $\operatorname{supp} f \subseteq T$. Again, two cases are possible: a) $x \in L$ and b) $x \notin L$. In case a), as noted above, we have $(\psi_b f)(x) = e'$. We show that this is true also in case b). For this it suffices to check that $b \odot x \notin T$. But if $b \odot x \in T$, then $b^{-1} \cdot (b \odot x) \in S_0^2 \subseteq S_1$, whence $b \cdot x \in S_0^3 = S_1$. Now, we obtain $b^{-1} \cdot (b \odot x) = b^{-1} \odot (b \odot x) = x$, i.e., $x \in S_0^2 \subseteq S_1$, whence $b \cdot x \in S_0^3 = S_1$, i.e., $b \cdot x = b \odot x$. This contradicts the assumption $b \cdot x \notin T$. The approximability of φ on $\operatorname{Fin}(A, B)$ is proved.

In order to prove statement iii), we note that if a group C is finite, then, obviously, $A \operatorname{Wr} C = A \operatorname{wr} C$, and statement ii) shows that if A is an LEF-group, then so is $A \operatorname{Wr} C$.

Now, let B be locally finite, and let A be an LEF-group. For a finite set $F \subseteq A \operatorname{Wr} B$, $F = \{\langle b_1, f_1\rangle, \ldots, \langle b_n, f_n\rangle\}$, let $\langle F\rangle$ be the subgroup generated by F. We choose a finite set $S \subseteq B$ such that

$$\forall i, j \leq n \; f_i \neq f_j \to f_i|S \neq f_j|S,$$

and introduce the finite subgroup C generated by $S \cup \{b_1, \dots, b_n\}$. It is clear that if $\langle b, f\rangle \in F$, then $b \in C$. Obviously, the map $\varphi\colon \langle F\rangle \longrightarrow A \operatorname{Wr} C$ acting by the rule $\varphi(b, f) = \langle b, f|C\rangle$ is a homomorphism injective on F. Now statement iii) follows from the lemma, because, by the above remark, $A \operatorname{Wr} C$ is an LEF-group. □

Corollary 1. *An extension of an* LEF*-group by a locally finite group is an* LEF*-group.*

Proof. Any extension of A by B can be embedded into the Cartesian wreath product $A \operatorname{Wr} B$. □

Corollary 2. *The class of* LEF*-groups is not closed under semidirect product.*

Proof. As already mentioned, the class of LEF-groups is not closed under extension. Hence it is not closed under Cartesian wreath product either (see the proof of Corollary 1). But the Cartesian wreath product $A \operatorname{Wr} B$ is a semidirect product $A^B \leftthreetimes B$, and, by statement i) of the theorem, if A is an LEF-group, then so is A^B. □

Remark. It is well known that all central extensions of a group G by a group Π that correspond to any fixed action $\varphi\colon \Pi \longrightarrow \operatorname{Aut}(G)$ are determined (up to isomorphism) by the elements of the first cohomology group $H^1(\Pi, Z(G))$, where $Z(G)$ is the center of G. Since this group is an LEF-group, it is natural to expect that if G and Π are LEF-groups, and φ is an equivariantly approximable action, then the above-mentioned extensions are LEF-groups. So far, this is open.

§3. Freely approximable group actions

In this section we give a new characterization of LEF-groups, which deals with actions. First (in Subsections 1° and 2°) we consider the actions of groups on arbitrary sets; then (in Subsections 3°–5°) we treat the actions of groups on spaces with quasiinvariant measure. Only effective group actions are considered in what follows.(3)

1°. We start with showing that, in some weak sense, any action of an infinite group is approximable by actions of finite groups on finite sets.

Proposition. *Suppose that a group G acts on a set A. Then for any finite sets $K \subseteq G$ and $H \subseteq A$ there is a finite set L with $H \subseteq L \subseteq A$ and an injective map $\varphi\colon K \longrightarrow \operatorname{Sym}(L)$ such that*

$$\forall \xi \in K\ \forall x \in L\ (\xi \cdot x \in L \to \varphi\xi(x) = \xi \cdot x). \tag{6}$$

Proof. Since the action of G on A is effective and K is finite, we can find a finite set $B \subseteq A$ such that

$$\forall \xi, \eta \in K\ (\xi \neq \eta \to \exists x \in B\ (\xi \cdot x \neq \eta \cdot x)).$$

Putting $L = H \cup B \cup K \cdot B$, for any $x \in L$, $\xi \in G$ we define

$$h_\xi(x) = \min\{\, i \mid \xi^{-i}x \in L,\ \xi^{-(i+1)}x \notin L,\ i \in \mathbb{N}\,\}.$$

In general, it may happen that $h_\xi(x) = +\infty$, i.e., $\xi^{-i}x \in L$ for any $i \in \mathbb{N}$. It is easily seen that $\xi \cdot x \in L$ in this case. For any $\xi \in G$, we define a map $\varphi\xi\colon L \longrightarrow L$ by the rule

$$\varphi\xi(x) = \begin{cases} \xi \cdot x & \text{if } \xi \cdot x \in L; \\ \xi^{-h_\xi(x)}x & \text{if } \xi \cdot x \notin L. \end{cases}$$

(3)We recall that an action $\varphi\colon G \longrightarrow \operatorname{Sym}(A)$ is said to be effective if $\operatorname{Ker}\varphi = 0$.

It is not hard to show that if $\xi \in K$, then the map $\varphi\xi$ is injective. Thus, we have defined a map $\varphi\colon K \longrightarrow \mathrm{Sym}(L)$ satisfying (6). The injectivity of φ follows from the definition of the finite set L. $\square$

We say that a group G is *weakly approximable* by finite groups if the action of G on itself by left shifts satisfies the conditions of the above proposition. Thus, we have the following statement.

Theorem. *Every infinite group G is weakly approximable by finite groups.*

2°. If we require additionally that, under the assumptions of the preceding proposition, the subgroup generated by the set $\varphi(K) \subseteq \mathrm{Sym}(L)$ (we denote this subgroup by $\langle\varphi(K)\rangle$) acts freely([4]) on L, then we arrive at the notion of a *freely approximable* (FA) *action.* Below it will be shown that the class of groups possessing freely approximable action is the class of LEF-groups.

Definition. An effective action of a group G on a set A is called an FA-action if for any finite sets $K \subseteq G$ and $H \subseteq A$ there is a finite set L with $H \subseteq L \subseteq A$ and an injective map $\varphi\colon K \longrightarrow \mathrm{Sym}(L)$ such that condition (6) is fulfilled, and $\langle\varphi(K)\rangle$ acts freely on L.

Lemma 1. *Any* FA-*action of a group G is free.* $\square$

Theorem. *A group G has an* FA-*action if and only if G is an* LEF-*group. In this case, any free action of G is freely approximable.*

Proof. If G is an LEF-group, then the action of G on itself by left shifts is freely approximable. Indeed, let $K, H \subseteq G$, and let $H_1 = K \cup H \cup \{e\}$. By Corollary 3 to Theorem 2.1, there exists a finite group $(L, \odot)$ such that $H_1 \subseteq L \subseteq G$ and condition (3) is fulfilled. We define a map $\varphi\colon K \longrightarrow \mathrm{Sym}(L)$ by putting $\varphi(k)(l) = k \odot L$ for any $l \in L$. It is clear that φ is injective, (6) follows from (3), and $\langle\varphi(K)\rangle$ acts freely on L, because the action of a group on itself by left shifts is free. The "if" part of our theorem is proved.

Now, suppose that an LEF-group G acts freely on a set A. Then A splits into the orbits, and on each of them the action of G is free and transitive, i.e., equivalent to the action of G on itself by left shifts. This yields the last statement of the theorem.

To prove the "only if" part, we relax somewhat the condition of free approximability, namely, in the above definition we replace condition (6) by the following weaker one:

$$\forall\xi \in K\ \forall x \in H\ (\xi \cdot x \in H \rightarrow \varphi\xi(x) = \xi \cdot x). \tag{7}$$

In this case φ is said to be *weakly freely approximable* (WFA). Obviously, every FA-action is a WFA-action (actually, in what follows we shall see that the converse is also true). The following lemma is similar to Lemma 1.

Lemma 2. *Every effective* WFA-*action of a group G is free.* $\square$

Now, the "only if" part of Theorem 3.2 is a consequence of the following two lemmas.

Lemma 3. *A restriction of a* WFA-*action of a group G to any of its orbits is also a* WFA-*action.*

Proof. Since the group G is infinite (otherwise, the theorem is trivial) and the action is free, all orbits are also infinite. Fixing an orbit A', we consider the restriction of our action to A'. Let $K \subseteq G$, $H \subseteq A'$, $\#K, \#H < \infty$. We choose $L \subseteq A$ and $\varphi\colon K \longrightarrow \mathrm{Sym}(L)$ satisfying (7). Since A' is infinite we can find $B \subseteq A'$ such that $B \cap L = \varnothing$, $\#B = \#(L \setminus A')$. Let $L' = B \cup (L \cap A')$. Then $H \subseteq L' \subseteq A'$ and

([4])I.e., $\forall\xi \in \langle\varphi(K)\rangle\ \forall x \in L\ (\xi \neq e \rightarrow \varphi\xi(x) \neq x)$.

$\#L = \#L'$. We fix any bijection $\psi\colon L \longrightarrow L'$ identical on $L \cap A'$. If $\xi \in K$, then we define $\varphi'\xi\colon L' \longrightarrow L'$ by the relation $\varphi'\xi = \psi \circ \varphi\xi \circ \psi^{-1}$ (the symbol $\circ$ denotes the composition of maps). It is clear that the groups $\langle\varphi(K)\rangle$ and $\langle\varphi'(K)\rangle$ (and their actions on L and L', respectively) are isomorphic, i.e., $\langle\varphi'(K)\rangle$ acts freely on L'. It only remains to check that φ' satisfies condition (7) (precisely this part of the proof fails for condition (6)). If $\xi \in K$, $x \in H$, and $\xi \cdot x \in H$, then, since (7) is true for φ, we have $\varphi\xi = \xi \cdot x \in H \subseteq A'$, i.e., $x, \varphi\xi(x) \in L \cap A'$. Since ψ is identical on $L \cap A'$, we obtain $\varphi\xi(x) = \varphi'\xi(x)$, i.e., $\varphi'\xi(x) = \xi \cdot x$, which proves (7) for φ' and L'. □

Lemma 4. *If G has a transitive* WFA*-action, then G is an* LEF*-group.*

Proof. Any free transitive action is isomorphic to the action of a group on itself by left shifts; therefore, it suffices to prove that if the latter action is WFA, then G is an LEF-group.

Let $H \subseteq G$ be a finite set. Without loss of generality we may assume that $e \in H$. Since the action by left shifts is WFA, there exists a finite set $L' \supseteq K$ and an injective map $\varphi'\colon K \longrightarrow \mathrm{Sym}(L')$ satisfying (7) such that $\langle\varphi(K)\rangle$ acts freely on L'. Let L be the orbit of the element $e \in L'$. By (7), since $e \in H = K$, for any $\xi \in K$ we have $\xi \cdot e = \xi \in K$, i.e., $\varphi'\xi(e) = \xi \cdot e = \xi \in L$, whence $K \subseteq L$. We put $\varphi\xi = \varphi'\xi|L$ for all $\xi \in K$. Then $\varphi\xi \in \mathrm{Sym}(L)$, and the group $\langle\varphi(K)\rangle$ acts freely and transitively on L. Moreover, the map $\varphi\colon K \longrightarrow \mathrm{Sym}(L)$ is injective, because if $\xi \neq \eta$, then $\varphi\xi(e) = \varphi'\xi(e) = \xi \cdot e = \xi$, and $\varphi\eta(e) = \eta$. Since the action of $\langle\varphi(K)\rangle$ on L is free and transitive, the map $\psi\colon \langle\varphi(K)\rangle \longrightarrow L$ such that $\psi(g) = g(e)$ for all $g \in \langle\varphi(K)\rangle$ is a bijection, i.e., the operation $\odot\colon L^2 \longrightarrow L$ defined by the relation

$$l_1 \odot l_2 = \psi(\psi^{-1}(l_1) \circ \psi^{-1}(l_1)) \tag{8}$$

turns L into a group.

We check that the set H and the group $(L, \odot)$ meet the requirements of Definition 1.1. Let $h_1, h_2 \in H = K$ and $h_1 \cdot h_2 \in H$. Then, by (7), $\varphi h_1(h_2) = h_1 \cdot h_2$. If $h_1 = g(e)$, i.e., $\psi(h_1) = g$, and $h_2 = h(e)$, i.e., $\psi(h_2) = h$, then $h_1 \odot h_2 = g(h(e)) = g(h_2)$ by (8). Since $h_1 \in H$, we have $h_1 \cdot e = h_1 \in H = K \subseteq L$, i.e., $\varphi h_1(e) = h_1 \cdot e = h_1$ (by (7)). Hence, $\varphi h_1(e) = g(e)$; since $\varphi h_1, g$ are elements of the group $\langle\varphi(K)\rangle$ (which acts on L freely and transitively) and $e \in L$, we obtain $\varphi h_1 = g$. Thus, $h_1 \odot h_2 = \varphi(h_1)(h_2)) = h_1 \cdot h_2$ by (7). □

3°. In what follows we consider invariant and quasiinvariant actions of a countable group G on a measurable space (X, μ) with finite measure.

Definition. An action of a group G on a space (X, μ) is said to be *uniformly approximable* (UA) if for any $g_1, \dots, g_s \in G$ and any $\varepsilon > 0$ there exist invariant or quasiinvariant transformations $k_1, \dots, k_s$ of the space (X, μ) such that they generate a finite group and

$$\mu\{\, x \in X \mid g_i x = k_i x,\ i = 1, \dots, s \,\} > 1 - \varepsilon.$$

If, moreover, the action of the group K generated by $\{k_1, \dots, k_s\}$ is free on (X, μ), then the action of G is said to be *uniformly freely approximable* (UFA).([5])

([5])We recall that an invariant or a quasiinvariant action of a group G on a space (X, μ) is called *free* if

$$\forall g \in G\ (g \neq e \rightarrow \mu\{\, x \in X \mid gx = x \,\} = 0).$$

If we additionally require that (X,μ) be a Lebesgue space and $k_1,\dots,k_s$ belong to the Dye completion $[G]$ of G, then the definition of a UA-action is equivalent to the definition of a tame action well known in ergodic theory (see, e.g., [3, 4]). It is known that any group having an invariant tame action is amenable, and, conversely, (the Connes–Feldman–Weiss theorem) any invariant or quasiinvariant action of an amenable group G on a Lebesgue space is tame. Also, it is known that any countable group has some tame (i.e., UA) action. The notion of a UFA-action was introduced in [2], where a certain class of groups having such an action was described. In [8], it was proved that any finitely representable group admitting a UFA-action is residually finite. In essence, [8] contains a stronger statement: any group having a UFA-action is an LEF-group. Actually, the following statement is true.

Theorem. *A countable group G has a* UFA*-action on some measurable space if and only if G is an* LEF*-group.*

Proof. As already mentioned, the proof of the "only if" part is contained in [8]. To prove the "if" part, on the countable group G we define a probability distribution such that the probability of each element be positive. It is clear that in this case the action of G on itself by left shifts is quasiinvariant with respect to a measure equivalent to Haar measure.

Let $g_1,\dots,g_s\in G$, and let $\varepsilon>0$. We enlarge $\{g_1,\dots,g_s\}$ up to a finite set $H\ni e$ such that $\mu(H)>1-\varepsilon$. By Lemma 1.1, there exists a finite group $(L,\odot)$ such that $H^2\subseteq L\subseteq G$ and $h_1\cdot h_2=h_1\odot h_2$ for all $h_1,h_2\in H$.

We split $G\setminus L$ into a countable union $\bigcup_{\nu=1}^{\infty}L_\nu$ of pairwise disjoint sets L_ν of cardinality equal to that of L, and, for each ν, fix a bijection $\varphi_\nu\colon L_\nu\longrightarrow L$. Also, $L_0=L$, and let $\varphi_0\colon L_0\longrightarrow L$ be the identity map. For $i\le s$, we define $k_i\colon G\longrightarrow G$ by the relation $k_i(g)=\varphi_\nu^{-1}(g_i\odot\varphi_\nu(g))$, where $g\in L_\nu$, $\nu\le 0$. Obviously, the group K generated by the transformations k_i, $i=1,\dots,s$, is isomorphic to the subgroup of $(L,\odot)$ generated by the elements g_i. Its action on G is free, because the action of $(L,\odot)$ on itself by left shifts is free. Since $g_i\in H\subseteq L$ for $i<s$, we have $g_i\cdot h=g_i\odot h=k_i(h)$, for all $h\in H$, i.e., $\mu(B)\ge 1-\varepsilon$. □

Remarks. 1. In this proof we have constructed an action of an LEF-group G on a space with discrete measure. Using this action, it is easy to construct a UFA-action of this group on some space with continuous measure.

2. It is easily seen that any UFA-action of a group G is free. By analogy with the last statement of Theorem 3.2, it is natural to ask whether any action of an LEF-group G with quasiinvariant finite measure is a UFA-action. Keeping in the mind the Connes–Feldman–Weiss theorem mentioned above, it is reasonable to conjecture that every free quasiinvariant action of an amenable LEF-group on any Lebesgue space with finite measure is a UFA action.

3. For finitely generated groups, a proof of the above theorem is contained also in the papers [9, 10].

4°. Let $\mathfrak{A}(X,\mu)$ be the group of all transformations of a space X. The metric $d(f,g)=\mu\{\,x\mid f(x)=g(x)\,\}$ defines a uniform topology on $\mathfrak{A}(X,\mu)$. The uniform approximability of an action of a group G on (X,μ), treated in the preceding subsection, is approximability with respect to this topology.

On $\mathfrak{A}(X,\mu)$ it is possible to consider a weak topology defined by the base of neighborhoods

$$\mathcal{T}=\{\,V(A_1,\dots,A_n,\varepsilon)\mid A_i\subseteq X,\ \mu(A_i)>0,\ \varepsilon>0,\ n\in\mathbb{N}\,\},$$

where $V(A_1,\dots,A_n,\varepsilon)=\{\,T\in\mathfrak{A}(X,\mu)\mid \mu(TA_i\Delta A_i)<\varepsilon,\ i=1,\dots,n\,\}$.

Definition. A nonsingular action(6) of a group G on a space (X,μ) is said to be *weakly approximable* (*weakly freely approximable*) if for any neighborhood $V \in \mathcal{T}$ and any $g_1,\dots,g_n \in G$ there exist nonsingular transformations $k_1,\dots,k_n$ of the space X such that the group generated by these transformations is finite (respectively, acts in X freely), and $g_i \cdot k_i^{-1} \in V$ for all $i \leq n$.

It is easily seen that any UA- (UFA-)action is also weakly (freely) approximable.

By analogy with the foregoing it is natural to ask the following questions.

1) Does an arbitrary group G possess any weakly approximable invariant action? (As already mentioned, every countable group possesses a nonsingular UA-action.)

2) Is every free action of an LEF-group weakly freely approximable? Theorem 3.3 implies that every countable LEF-group has some weakly freely approximable action.

3) Is it true that a group G having a nonsingular weakly freely approximable action is an LEF-group?

5°. These questions can be generalized in the spirit of the definitions from [2].

Definition. A topological group $\mathfrak{A}$ is said to be *abundant* if for any neighborhood V of the identity element of $\mathfrak{A}$ and any finite subset $\{g_1,\dots,g_n\}$ of this group there exist $k_1,\dots,k_n \in \mathfrak{A}$ such that

1. the subgroup of $\mathfrak{A}$ generated by $k_1,\dots,k_n$ is finite;
2. $g_i \cdot k_i^{-1} \in V$, $i = 1,\dots,n$.

The first of the questions formulated above is equivalent to the question as to whether any group of transformations with invariant (quasiinvariant) measure is abundant in the weak topology.

It should be noted that the group of all unitary operators on a Hilbert space is obviously abundant in the weak topology, but is not in the uniform topology (indeed, if a group generated, e.g., by two unitary operators is uniformly approximable by finite groups, then this group is amenable; see [4] and the references therein).

A similar definition can be formulated for algebras: a topological algebra $\mathcal{A}$ is said to be *abundant* if for any neighborhood V of zero and any finite subset $\{a_1,\dots,a_n\}$ of $\mathcal{A}$ there exists a finite-dimensional subalgebra of $\mathcal{A}$ and a finite subset $\{a_1',\dots,a_n'\}$ of this subalgebra such that $a_i - a_i' \in V$ for all $i \leq n$. This property is also related to amenability and nuclearity of algebras.

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(6)I.e., an action with invariant or quasiinvariant finite measure.

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