1 $|f|$ and f^2 are integrable when f is integrable

Lemma 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. Then for each $i \in \{1, 2, \ldots, n\}$, $M_i(f) - m_i(f) = \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}$.*

Proof. Let $x, y \in [x_{i-1}, x_i]$. Without loss of generality assume that $f(x) \geq f(y)$ and observe that $M_i(f) \geq f(x)$ and $m_i(f) \leq f(y)$. These inequalities imply that $M_i(f) - m_i(f) \geq f(x) - f(y)$. It now follows that

$$M_i(f) - m_i(f) \geq \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}. \tag{1}$$

Let $\epsilon > 0$ be given. There exist $x, y \in [x_{i-1}, x_i]$ such that $f(x) > M_i(f) - \frac{\epsilon}{2}$ and $f(y) < m_i(f) + \frac{\epsilon}{2}$. So $f(x) - f(y) > M_i(f) - m_i(f) - \epsilon$, and therefore, $|f(x) - f(y)| > M_i(f) - m_i(f) - \epsilon$. It now follows that $\sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\} > M_i(f) - m_i(f) - \epsilon$. Since this holds for any $\epsilon > 0$, we have

$$\sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\} \geq M_i(f) - m_i(f). \tag{2}$$

The inequalities (1) and (2) imply the desired equality. □

Theorem 1.2. *Suppose that $f : [a, b] \to \mathbb{R}$ is an integrable function. Then $|f|$ is also integrable on $[a, b]$.*

Proof. Let $\epsilon > 0$ be given. Since f is integrable, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. For any $i \in \{1, 2, \ldots, n\}$ and all $x, y \in [x_{i-1}, x_i]$, we have $|\,|f(x)| - |f(y)|\,| \leq |f(x) - f(y)|$. So $M_i(f) - m_i(f)$ is an upper bound for the set $\{|\,|f(x)| - |f(y)|\,| : x, y \in [x_{i-1}, x_i]\}$, which implies that $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$. Now,

$$\begin{aligned} U(|f|, P) - L(|f|, P) = \sum_{i=1}^{n} (M_i(|f|) - m_i(|f|))\Delta x_i &\leq \sum_{i=1}^{n} (M_i(f) - m_i(f))\Delta x_i \\ &= U(f, P) - L(f, P) < \epsilon. \end{aligned}$$

This shows that $|f|$ is integrable on $[a, b]$. □

Theorem 1.3. *Suppose that $f : [a, b] \to \mathbb{R}$ is an integrable function. Then f^2 is also integrable on $[a, b]$.*

Proof. Since f is bounded on $[a, b]$, there exists a $B > 0$ such that $|f(x) + f(y)| < B$ for all $x, y \in [a, b.]$ Now let $\epsilon > 0$ be given. Since f is integrable, there exists a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that $U(f, P) - L(f, P) < \frac{\epsilon}{B}$. For any $i \in \{1, 2, \ldots, n\}$ and all $x, y \in [x_{i-1}, x_i]$, we have $|\,(f(x))^2 - (f(y))^2\,| = |f(x) + f(y)|\,|f(x) - f(y)| < B|f(x) - f(y)|$. So $B(M_i(f) - m_i(f))$ is an upper bound for the set $\{|\,(f(x))^2 - (f(y))^2\,| : x, y \in [x_{i-1}, x_i]\}$, which implies that $M_i(f^2) - m_i(f^2) \leq B(M_i(f) - m_i(f))$. Now,

$$\begin{aligned} U(f^2, P) - L(f^2, P) = \sum_{i=1}^{n} (M_i(f^2) - m_i(f^2))\Delta x_i &\leq \sum_{i=1}^{n} B(M_i(f) - m_i(f))\Delta x_i \\ &= B(U(f, P) - L(f, P)) < B\frac{\epsilon}{B} = \epsilon. \end{aligned}$$

This shows that f^2 is integrable on $[a, b]$. □

2 Integration for continuous function

Theorem 2.1. *Let* $f : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and let* $P_n = \{x_0 = a, x_1 = a + \frac{(b-a)}{n}, x_2 = a + 2\frac{(b-a)}{n}, \ldots, x_n = b\}$. *Then* $\displaystyle\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n)$.

Proof. It suffices to show that $\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$ since exercise 29.5 in [1] will then imply the result. Let $\epsilon > 0$ be given. Since f is uniformly continuous on $[a, b]$, there exists a $\delta > 0$ such that when $|x - y| < \delta$, $|f(x) - f(y)| < \frac{\epsilon}{b-a}$. Also, continuity of f implies that for each $i \in \{1, 2, \ldots, n\}$, there exist points $s_i, t_i \in [x_{i-1}, x_i]$ such that $M_i = f(t_i)$ and $m_i = f(s_i)$. Now if $\Delta x_i = \frac{(b-a)}{n} < \delta$,

$$\begin{aligned} U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i = \sum_{i=1}^{n}(f(t_i) - f(s_i))\Delta x_i < \frac{\epsilon}{b-a}\sum_{i=1}^{n}\Delta x_i \\ = \frac{\epsilon}{b-a}(b-a) = \epsilon. \end{aligned}$$

Since $\lim_{n\to\infty}\frac{(b-a)}{n} = 0$, there exists a $N \in \mathbb{R}$ such that when $n > N$, we have $\frac{(b-a)}{n} < \delta$. So when $n > N$, we get $U(f, P_n) - L(f, P_n) < \epsilon$, which implies that $\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$. □

Corollary 2.2. *Suppose that* $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$. *Let* $P_n = \{x_0 = a, x_1 = a + \frac{(b-a)}{n}, x_2 = a + 2\frac{(b-a)}{n}, \ldots, x_n = b\}$ *and for each* $i \in \{1, 2, \ldots, n\}$, *let* $x_i^* \in [x_{i-1}, x_i]$ *be sample points. Then* $\displaystyle\int_a^b f = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x_i$.

Proof. For each $i \in \{1, 2, \ldots, n\}$, $m_i \leq f(x_i^*) \leq M_i$. So

$$L(f, P_n) = \sum_{i=1}^{n} m_i\,\Delta x_i \leq \sum_{i=1}^{n} f(x_i^*)\Delta x_i \leq \sum_{i=1}^{n} M_i\,\Delta x_i = U(f, P_n).$$

Since $\lim U(f, P_n) = \lim L(f, P_n)$, the Squeeze Theorem implies that

$$\int_a^b f = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x_i = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n).$$

□

References

[1] S. Lay, *Analysis with an introduction to proof*, Prentice Hall, Inc., Englewood Cliffs, NJ, 1986.