Solutions
Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Augustana College, April 9, 1999

1. *Find all solutions to the following system of equations where $x_1, x_2, \ldots x_n$ are positive real numbers.*

$$\begin{aligned} x_1 &= x_1x_2 + \frac{1}{4} \\ x_2 &= x_2x_3 + \frac{1}{4} \\ &\vdots \\ x_{n-1} &= x_{n-1}x_n + \frac{1}{4} \\ x_n &= x_nx_1 + \frac{1}{4} \end{aligned}$$

Solution: Notice that the first equation can be written as $x_1(1-x_2) = \frac{1}{4}$. Now, $y = x(1-x)$ is a parabola opening downward. Hence its maximum value occurs at the vertex, which is halfway between the intercepts 0 and 1. In other words, $x(1-x) \leq \frac{1}{4}$, for all x.

Suppose that $x_1, x_2, \ldots x_n$ satisfy the given equations. Then $x_1(1-x_2) = \frac{1}{4}$, $x_2(1-x_3) = \frac{1}{4}$, and $x_n(1-x_1) = \frac{1}{4}$. Multiplying these equations together and rearranging terms gives

$$x_1(1-x_1)x_2(1-x_2)\cdots x_n(1-x_n) = \left(\frac{1}{4}\right)^n.$$

Therefore, if $x_i(1-x_i) < \frac{1}{4}$, for any i,

$$x_1(1-x_1)x_2(1-x_2)\cdots x_n(1-x_n) < \left(\frac{1}{4}\right)^n.$$

Hence $x_i(1-x_i) = \frac{1}{4}$, for all i. Since the parabola $y = x(1-x)$, attains its maximum only at one point, $x_i = \frac{1}{2}$, for all i.

This problem can also be done by observing that the terms are defined in terms of the function $f(x) = \frac{x-1/4}{x}$ where $x_i = f(x_{i-1})$, the subscripts being read modulo n. Since $x > 0$, this function is increasing and it is possible to show that the only solution is for $x_1 = x_2 = \cdots = x_n$.

2. *Evaluate the sum:* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

Solution: By the theory of partial fractions,

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}.$$

Multiplying both sides of this equation by n and then letting $n = 0$ gives $\frac{1}{2} = A$. Multiplying both sides by $n+1$ and then letting $n = -1$ gives $\frac{1}{(-1)\cdot 1} = -1 = B$. Multiplying both sides by $n+2$ and then letting $n = -2$ gives $\frac{1}{(-2)\cdot(-1)} = \frac{1}{2} = C$. Hence,

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}.$$

Since the three series whose terms make up the series on the right hand side of this equation are divergent, the summation on the right cannot be broken into three summations. However, for N large,

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2} &= \left(1 - \frac{2}{2} + \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) \\
&+ \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \cdots + \left(\frac{1}{N-2} - \frac{2}{N-1} + \frac{1}{N}\right) \\
&+ \left(\frac{1}{N-1} - \frac{2}{N} + \frac{1}{N+1}\right) + \left(\frac{1}{N} - \frac{2}{N+1} + \frac{1}{N+2}\right) \\
&= 1 - \frac{1}{2} - \frac{1}{N+1} + \frac{1}{N+2} = \frac{1}{2} - \frac{1}{(N+1)(N+2)}
\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)(n+2)} = \lim_{n\to\infty} \left(\frac{1}{2} - \frac{1}{(N+1)(N+2)}\right) = \frac{1}{2}.$$

Therefore, the sum of the given series is $\frac{1}{4}$.

Note: This sum can also be found by expressing the given series in terms of two converging series and then changing the order of summation. The first step of this solution is to observe that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \sum_{n=1}^{\infty} \frac{1/2}{n(n+1)} - \frac{1/2}{(n+1)(n+2)}.$$

3. (a) *Show that if a, b, c, d are positive integers such that $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.*

Solution: Now,

$$\frac{a}{b} < \frac{a+c}{b+d} \iff a(b+d) < b(a+c) \iff ad < bc \iff \frac{a}{b} < \frac{c}{d}.$$

Likewise,

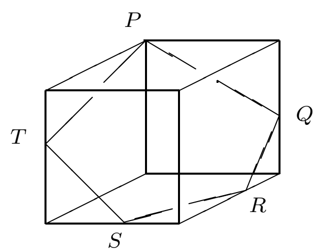
$$\frac{a+c}{b+d} < \frac{c}{d} \iff (a+c)d < (b+d)c \iff ad < bc \iff \frac{a}{b} < \frac{c}{d}.$$

Therefore, if $\frac{a}{b} < \frac{c}{d}$, then the desired inequality holds.

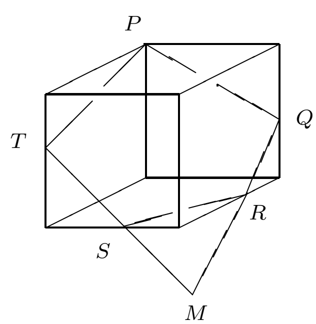
(b) *Find the largest rational number $\frac{p}{q}$ such that $\frac{p}{q} < \frac{2}{5}$, where $q < 100$.*

Solution: Suppose $\frac{p}{q}$ is the largest fraction with the desired properties. Since $\frac{p+2}{q+5} < \frac{2}{5}$, by part a), we must have that $q + 5 \geq 100$. Therefore, $q = 95, 95, 97, 98$, or 99.
If $q = 95$, then $\frac{p}{95} < \frac{2}{5}$ implies $p < \frac{2}{5} \cdot 95 = 38$. Hence, by the maximality of $\frac{p}{q}$, we must have $p = 37$, in this case. If $q = 96$, then $\frac{p}{96} < \frac{2}{5}$ implies $p < \frac{2}{5} \cdot 96 = 38 + \frac{2}{5}$ and thus $p = 38$. If $q = 97$, $\frac{p}{97} < \frac{2}{5}$ means that $p < \frac{2}{5} \cdot 97 = 38 + \frac{4}{5}$ and $p = 38$. If $q = 98$, $\frac{p}{98} < \frac{2}{5}$ means $p < 38 + \frac{6}{5}$ which implies $p = 39$. Finally, if $q = 99$, $\frac{p}{99} < \frac{2}{5}$ forces $p < 38 + \frac{8}{5}$ and $p = 39$.
Thus the only fractions that need be considered are $\frac{37}{95}, \frac{38}{96}, \frac{38}{97}, \frac{39}{98}$, and $\frac{39}{99}$. Since $\frac{37}{95} < \frac{1}{1}$, part a) implies that $\frac{37}{95} < \frac{38}{96}$. Clearly, $\frac{38}{96} > \frac{38}{97}$. Since $\frac{38}{96}$ and $\frac{39}{99}$ can be reduced to fractions with denominators less than 95, they and all fractions smaller than either of them can be eliminated. Therefore, by elimination, the fraction with the desired property is $\frac{39}{98}$.

4. *The point P is a vertex of a cube. Consider a plane which passes through P and intersects the opposite face in a line. Let Q, R, S, T be the points where this plane intersects the edges of the cube, as shown below. Show that $TP + PQ > QR + RS + ST$.*



Solution: Note that PT and QR are on opposite sides of a cube and are thus parallel. Similarly, $PQ \parallel TS$. Let M be the intersection of QR extended and TS extended. This point exists since it is the intersection of the plane of the face containing QR, the plane of the face containing TS, and the plane which determines the pentagon $PQRST$. The quadrilateral $PQMT$ is a parallelogram since opposite sides are parallel.

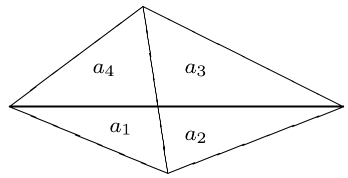


By the triangle inequality, $SR < SM + MR$. But $SM = TM - TS = PQ - TS$ and $MR = QM - QR = PT - QR$, since opposite sides of a parallelogram are equal. Therefore, by substitution,

$$SR < (PQ - TS) + (PT - QR)$$

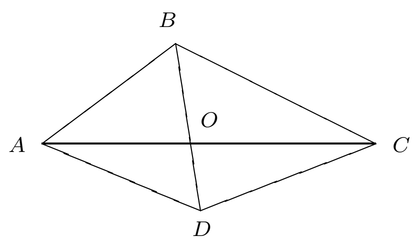
which implies $SR + TS + QR < PQ + PT$, as required.

5. *The diagonals of a quadrilateral divide it into four triangles having areas* a_1, a_2, a_3, a_4 *as shown.*



   (a) *Show* $a_1 a_3 = a_2 a_4$.

   Solution: Let O be the point where the diagonals intersect and let A, B, C, D be the vertices of the quadrilateral as shown.



   Note that $\triangle AOB$ and $\triangle BOC$ have the same height. Thus their areas are in the same proportion as the lengths of their bases, i.e. $a_3/a_4 = OC/OA$. Similarly, by considering $\triangle AOD$ and $\triangle DOC$, we have that $a_2/a_1 = OC/OA$. Hence $a_3/a_4 = a_2/a_1$ and the result follows.

   (b) *If the area of all four triangles are integral and the areas of three of them are successive integers, what is the area of the fourth?*

   Solution: Let the three integers in question be n, $n+1$, and $n+2$. Let m be the integral area of the fourth triangle. There are three cases to consider:

   Suppose that the triangles of areas m and $n+2$ are opposite. Then $m(n+2) = n(n+1)$ and $m = \frac{n^2+n}{n+2} = n - \frac{n}{n+2}$, which is impossible since m and n are integers.

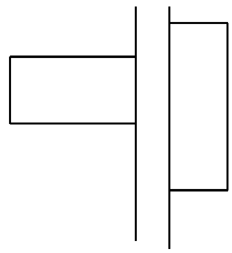
   Suppose that the triangles of area m and $n+1$ are opposite, then $m(n+1) = n(n+2)$ and $m = \frac{n^2+2n}{n+1} = n + \frac{n}{n+1}$, which is also impossible.

   Finally suppose the triangles of areas m and n are opposite, then $mn = (n+1)(n+2)$ and $m = \frac{n^2+3n+2}{n} = n + 3 + \frac{2}{n}$. Therefore n is either 1 or 2. In both cases, we have that m, the area of the fourth triangle, is 6.

   Note: There do exist examples where $n = 2$ and $m = 6$ as well as examples where $n = 1$ and $m = 6$. To find an example with $n = 2$ and $m = 6$, construct a line 4 units long and draw two lines parallel to it, one on each side of the segment, one 2 units above the segment and one 4 units below the segment. Let B be any any point on the upper parallel line, A and C the endpoints of the line segment and O a point on the segment so that $A0 = 1$ and $OC = 3$. Letting D be the intersection of $B0$ and the lower line gives a quadrilateral $ABCD$ with the desired properties.

6. *Running through a farmer's field is a straight river. He wants to use 160 feet of fencing to enclose either one or two rectangular fields. He can use the river as one side of the field and*

if he encloses two fields he can make one field on each side of the river, as shown. What should the dimensions of the field(s) be so that the total area enclosed is a maximum?



Solution: We begin with the following lemma: *If a rectangular field is enclosed on three sides using s feet of fence, then the largest possible area for the field is $\frac{s^2}{8}$.* Let x be the length of fence on each of the two, fenced, parallel sides and y be the length of the fence on the other side. Then $y = s - 2x$ and the area, A, of the field is given by $A = x(s - 2x)$. The graph of A is a parabola opening downward and thus its maximum value occurs at the vertex, midway between the intercepts, i.e. when $x = \frac{s}{4}$, $y = \frac{s}{2}$, $A = \frac{s^2}{8}$.

Now consider the problem of fencing possibly two fields using a river as a boundary of both fields. Let s be the amount of fencing used on one side of the river and $160 - s$ the amount used on the other, where $0 \leq s \leq 160$. By the lemma, the maximum area of the field on the one side is $\frac{s^2}{8}$ and on the other the maximum area is $\frac{(160-s)^2}{8}$. Thus the maximum possible area is the maximum value of $T = \frac{s^2}{8} + \frac{(160-s)^2}{8}$. The graph of T is a parabola, but it open upwards! Hence the maximum possible value for T occurs at the endpoints, i.e. when $s = 0$ or $s = 160$. Therefore, the dimensions of the field which gives the maximum area is $40' \times 80'$.