Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Augustana College, April 9, 1999

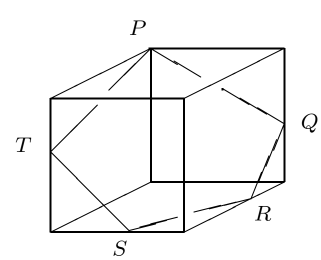
1. Find all solutions to the following system of equations where $x_1, x_2, \ldots x_n$ are positive real numbers.

$$\begin{aligned} x_1 &= x_1x_2 + \frac{1}{4} \\ x_2 &= x_2x_3 + \frac{1}{4} \\ &\vdots \\ x_{n-1} &= x_{n-1}x_n + \frac{1}{4} \\ x_n &= x_nx_1 + \frac{1}{4} \end{aligned}$$

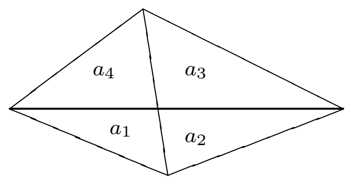
2. Evaluate the sum: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$.

3. (a) Show that if a, b, c, d are positive integers such that $\frac{a}{b} < \frac{c}{d}$, then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.
   (b) Find the largest rational number $\frac{p}{q}$ such that $\frac{p}{q} < \frac{2}{5}$, where $q < 100$.

4. The point P is a vertex of a cube. Consider a plane which passes through P and intersects the opposite face in a line. Let Q, R, S, T be the points where this plane intersects the edges of the cube, as shown below. Show that $TP + PQ > QR + RS + ST$.



5. The diagonals of a quadrilateral divide it into four triangles having areas a_1, a_2, a_3, a_4 as shown.



   (a) Show $a_1 a_3 = a_2 a_4$.

   (b) If the area of all four triangles are integral and the areas of three of them are successive integers, what is the area of the fourth?

6. Running through a farmer's field is a straight river. He wants to use 160 feet of fencing to enclose either one or two rectangular fields. He can use the river as one side of the field and if he encloses two fields he can make one field on each side of the river, as shown. What should the dimensions of the field(s) be so that the total area enclosed is a maximum?