Solutions
Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
McKendree College, March 27, 1998

Do any four of the six problems. Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution. Enjoy the problem solving.

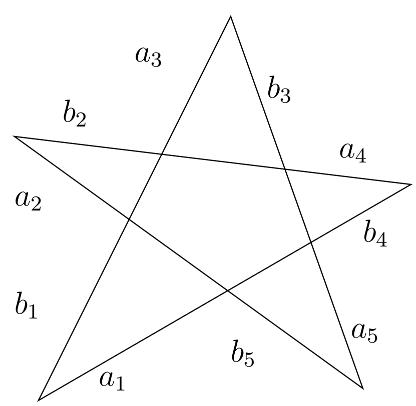
1. *For x and y positive integers, let $f(x,y)$ denote the minimum element in $\{x, y, (\frac{1}{x}+\frac{1}{y})\}$. Determine the maximum value of $f(x,y)$.*

   Solution: We show the maximum value is $\sqrt{2}$. If either $x \le \sqrt{2}$ or $y \le \sqrt{2}$, then $\frac{1}{x}+\frac{1}{y} \ge \sqrt{2}$. In this case,
$$f(x,y) = \min(x,y) \le \sqrt{2}.$$
   If $x > \sqrt{2}$ and $y > \sqrt{2}$, then
$$\frac{1}{x}+\frac{1}{y} < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$$
   Therefore, in any case, $f(x,y) \le \sqrt{2}$. Since $f(\sqrt{2},\sqrt{2}) = \sqrt{2}$, the maximum is $\sqrt{2}$.

2. *Let $a = \dfrac{888\ldots89 \times 333\ldots34}{666\ldots67 \times 444\ldots45}$ and $b = 1 + \frac{1}{2}\left(\frac{1}{333\ldots34} - \frac{1}{444\ldots45}\right)$, where there are 100 digits in each of $333\ldots34$, $444\ldots45$, $666\ldots67$, and $888\ldots89$. Which is larger, a or b?*

   Solution: a is larger. Let $M = 888\ldots89$, $m = 444\ldots45$, $N = 666\ldots67$, and $n = 333\ldots34$. We have $M = 2m-1$ and $N = 2n-1$. Hence
$$a = \frac{\frac{M}{m}}{\frac{N}{n}} = \frac{\frac{2m-1}{m}}{\frac{2n-1}{n}} = \frac{2-\frac{1}{m}}{2-\frac{1}{m}} > 1 + \frac{\frac{1}{n}-\frac{1}{m}}{2-\frac{1}{n}} > 1 + \frac{\frac{1}{n}-\frac{1}{m}}{2} = b.$$

3. *Five straight segments are joined to form a five-pointed star. The boundary of the star is formed by 10 line segments, whose lengths are a_i, b_i, $i = 1, 2, \ldots, 5$, as shown below. Prove that if $a_1 > b_1$, then there is a $k \in \{2,3,4,5\}$ with $a_k < b_k$.*



Solution: Assume, by way of contradiction, that $a_k \geq b_k$ for $k = 2, 3, 4, 5$. Consider the five triangles which make up the points of the star and let α_i, $i = 1, 2, \ldots, 5$, be the angle opposite a_i. Similiarly, let β_i, $i = 1, 2, \ldots, 5$, be the angle opposite b_i. In a triangle, if one side, say a, is larger than another, b, then the angle opposite a is larger than the angle opposite b. Thus our assumption implies $\alpha_k \geq \beta_k$, $k = 2, 3, 4, 5$. Also $\alpha_1 > \beta_1$. Since $\alpha_1 = \beta_2$, $\alpha_2 = \beta_3$, $\alpha_3 = \beta_4$, $\alpha_4 = \beta_5$, and $\alpha_5 = \beta_1$,

$$\alpha_1 > \alpha_5 \geq \alpha_4 \geq \alpha_3 \geq \alpha_2 \geq \alpha_1.$$

This contradition establishes the result.

4. *Find all real numbers* $x_1, x_2, \ldots, x_{1998}$ *such that*

$$\begin{aligned} x_1 x_2 &= x_1 - x_2 \\ x_2 x_3 &= x_2 - x_3 \\ &\cdots \\ x_{1997} x_{1998} &= x_{1997} - x_{1998} \\ x_{1998} x_1 &= x_{1998} - x_1 \end{aligned}$$

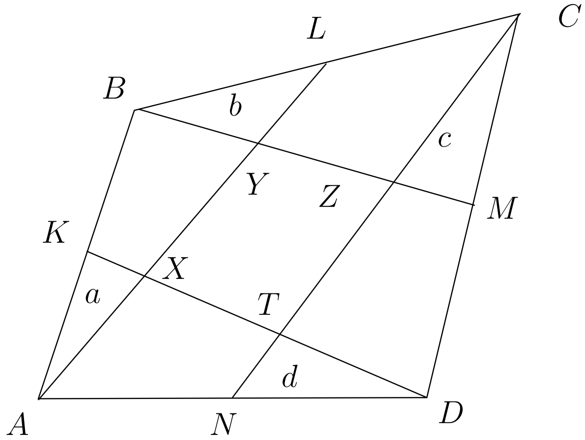
Solution: We show that the only solution is when $x_i = 0$ for all i. Observe that if $x_k = 0$ for some k, then x_{k+1}, where the subscripts are read mod 1998, is also zero. Hence if one is zero, they all are.

Suppose then that they are all non-zero. Dividing the first equation by $x_1 x_2$, the second by $x_2 x_3$ and so on yields

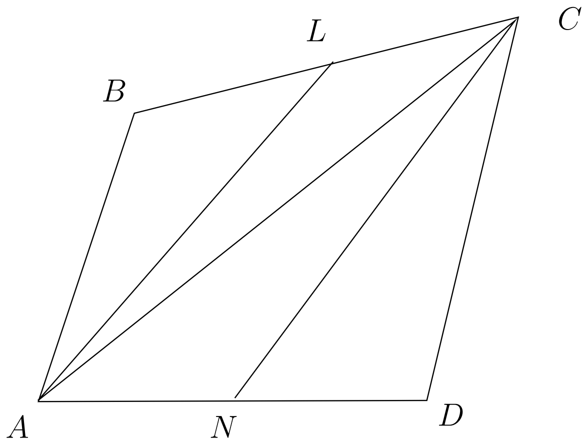
$$\begin{aligned} 1 &= \frac{1}{x_2} - \frac{1}{x_1} \\ 1 &= \frac{1}{x_3} - \frac{1}{x_2} \\ &\cdots \\ 1 &= \frac{1}{x_{1998}} - \frac{1}{x_{1997}} \\ 1 &= \frac{1}{x_1} - \frac{1}{x_{1998}} \end{aligned}$$

Adding these equations yields the contradictory result $1998 = 0$. Hence $x_1 = x_2 = \cdots = x_{1998} = 0$.

5. *Let* A, B, C, D *be the vertices of a convex quadrilateral;* K, L, M, N *the midpoints of the sides;* X, Y, Z, T *intersections of the lines joining vertices to midpoints of opposite sides; and* a, b, c, d *areas of portions of the quadrilateral as shown in the figure below. Find the area of quadrilateral* $XYZT$ *in terms of* a, b, c, *and* d.



Solution: Consider the original quadrilateral and draw lines LA, AC, and CN as shown:



Now Area($\triangle ABL$) = Area($\triangle ACL$) and Area($\triangle ACN$) = Area($\triangle DCN$), since the median divides the area of a triangle into two equal parts. It follows that

$$\text{Area}(\triangle ABL) + \text{Area}(\triangle DCN) = \frac{1}{2}\text{Area}(ABCD).$$

Similarly, by drawing lines MB, BD and DK,

$$\text{Area}(\triangle BCM) + \text{Area}(\triangle DAK) = \frac{1}{2}\text{Area}(ABCD).$$

Hence,

$$\text{Area}(\triangle ABL) + \text{Area}(\triangle DCN) + \text{Area}(\triangle BCM) + \text{Area}(\triangle DAK) = \text{Area}(ABCD).$$

But these four triangles cover all of $ABCD$ except for $XYZT$ and each of the areas a, b, c, and d is inside two of the triangles. Hence the sum of the areas of the four triangles is

$$\text{Area}(ABCD) - \text{Area}(XYZY) + a + b + c + d.$$

Hence, Area($XYZY$) $= a + b + c + d$.

6. *The graph of a non-negative, differentiable function* f *divides the triangle with vertices* $(0,0)$, $(0,x)$, *and* $(x, f(x))$ *into two parts having equal areas for each positive value of* x. *Find an explicit expression for* $f(x)$ *if* $f(1998) = 1998$.

   Solution: The area under $y = f(x)$ is half of the area of the triangle. Therefore,

$$\int_0^x = \frac{1}{2}\left(\frac{1}{2}xf(x)\right).$$

   Differentiating this expression by using the Fundamental Theorem of Calculus yields,

$$f(x) = \frac{1}{4}(f(x) + xf'(x)).$$

   This gives the differential equation $y = \frac{1}{4}(y + xy')$ or $3y = xy'$. This equation can be solved by separation of varialbles to give $y = cx^3$, for some constant c. Finally, using the initial condition gives $f(x) = x^3/(1998)^2$.