Solutions
Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Rockford College, March 21, 1997

Do any four of the six problems. Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution. Enjoy the problem solving.

1. *Jonathan loves candy bars. Each week, from 12:00 midnight Sunday until 12:00 midnight the following day he eats precisely 13 candy bars. Each day he eats at least one candy bar. Show if he continues this process forever there will eventually be a period of consecutive days during which he eats exactly eight candy bars.*

   Solution: Suppose Jonathan eats candy bars for n weeks. Number the days consecutively from 1 to $7n$ and, for $i = 1, 2, \ldots, 7n$, let a_i denote the number of candy bars eaten from day 1 to day i, inclusive. Since he eats at least one candy bar each day and exactly 13 each week,

   $$1 \leq a_1 < a_2 < \cdots < a_{7n} = 13n.$$

   For $i = 1, 2, \ldots, 7n$, let $b_i = a_i + 8$. Clearly,

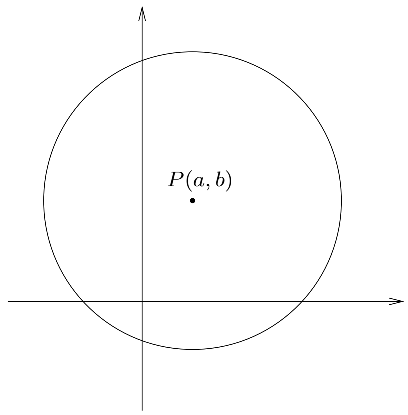
   $$9 \leq b_1 < b_2 < \cdots < b_{7n} = 13n + 8.$$

   Thus, there are $14n$, not necessarily distinct, numbers in the list

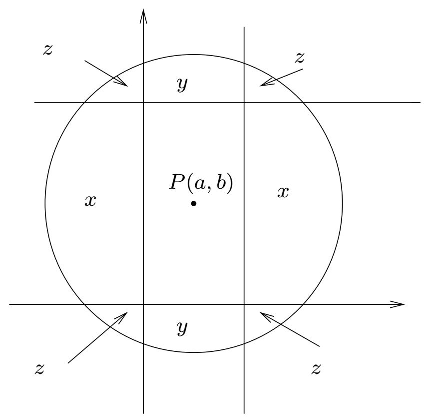
   $$a_1, a_2, \ldots, a_{7n}, b_1, b_2, \ldots, b_{7n}$$

   between 1 and $13n + 8$. Suppose $n > 8$. In this case, $14n > 13n + 8$ and two of the numbers in the list must be equal. Since $a_i \neq a_j$ and $b_i \neq b_j$, for $i \neq j$, it must be that $a_r = b_s$ for some r and s. That is, $a_r = b_s = a_s + 8$. Thus, exactly 8 candy bars are eaten in the period from day $s + 1$ to day r.

2. *Point $P(a, b)$ is in the first quadrant. A circle of radius R, center P, and containing points in each quadrant is drawn as shown below. If A_i is the area inside the circle and also the i-th quadrant, $i = 1, 2, 3, 4$, what is the value of $A_1 - A_2 + A_3 - A_4$?*



Solution: The coordinate axes together with the lines $x = 2a$ and $y = 2b$ divide the circle into 9 regions as shown below.



By symmetry, the two region labelled x are equal as are the two regions labelled y and the four regions labelled z. The remaining region is a $2a \times 2b$ rectangle. Using x, y, z to denote the area of the respective region gives

$$\begin{aligned} A_1 &= x + y + z + 4ab \\ A_2 &= x + z \\ A_3 &= z \\ A_4 &= y + z \end{aligned}$$

Hence, $A_1 - A_2 + A_3 - A_4 = 4ab$.

3. *You have a deck of cards and on each card there is a single digit between 0 and 0, inclusive. The digit on the top card equals the number of cards which have a one on them, and so forth until the digit on the last card is the number of cards with a nine on them. What are the digits, in order from top to bottom, on the cards?*

Solution: For $i = 0, 1, \ldots, 9$ let x_i denote the number of cards which have i on them. Since there are 10 cards,

$$x_0 + x_1 + \cdots + x_9 = 10.$$

Since each card is accounted for exactly once in the numbers on the cards, the sum of the numbers on the cards is also 10, i.e.

$$x_1 + 2x_2 + \cdots + 9x_9 = 10.$$

Subtracting these two equations and solving for x_0 gives

$$x_0 = x_2 + 2x_3 + \cdots + 8x_9. \qquad (*)$$

If $x_0 \leq 3$, equation (*) implies $x_5 = x_6 = x_7 = x_8 = x_9 = 0$. This means at least five of the cards have a zero on them which contradicts the assumption $x_0 \leq 3$. Hence $x_0 = j$ where $j \geq 4$. Since at least one card has a j on it, $x_j \geq 1$. If one of the x_i's for $i \geq 3$, $i \neq j$, is non-zero or if $x_j \geq 2$, then

$$x_2 + 2x_3 + \cdots + 8x_9 \geq j + 1.$$

Since the sum on the left equals $x_0 = j$, it must be that $x_i = 0$ for $i \geq 3, i \neq j$, and $x_j = 1$. Thus, $x_2 = 1$ from equation (*). Since there are only four non-zero digits on the cards, $x_0 = 6$. The desired number is 6210001000.

4. *Let f be a non-linear function which is differentiable on the interval $[a, b]$, where $a < b$. Show there is a number $c \in (a, b)$ such that*

$$f'(c) > \frac{f(b) - f(a)}{b - a}.$$

Solution: Let h be the function defined by

$$h(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a).$$

Since h is continuous on a closed interval it attains its extreme values there.

Suppose first the maximum of h is larger than $h(a)$. Let $c_1 \in (a, b)$ be such that $h(c_1)$ is the maximum value of h. Since h is differentiable on $[a, c_1]$, the Mean Value Theorem implies there is a point $c_2 \in (a, c_1)$, such that

$$h'(c_2) = \frac{h(c_1) - h(a)}{c_1 - a}.$$

Since $h(c_1) - h(a) > 0$, $h'(c_2) > 0$. As $h'(x) = f'(x) - \left(\frac{f(b)-f(a)}{b-a}\right)$, it follows that

$$f'(c_2) > \frac{f(b) - f(a)}{b - a}.$$

If the minimum of h occurs at c_3 where $h(c_3) < h(a)$, applying the Mean Value Theorem to $[c_3, b]$ and using the fact that $h(a) = h(b)$ yields a point $c_4 \in (c_3, b)$ such that

$$f'(c_4) > \frac{f(b) - f(a)}{b - a}.$$

It remains to consider the case $h(x)$ is constant on $[a, b]$. However in this case, $h'(x) = 0$ and f is linear.

5. *Find all triples (x, y, z) of positive real numbers such that $x^y = z, y^z = x$, and $z^x = y$.*

   Solution: Suppose (x, y, z) is a triple of positive real numbers which satisfies the equations above. Now,

$$x^{xyz} = (x^y)^{xz} = (z^x)^z = y^z = x.$$

   Hence $xyz = 1$. Suppose one x, y or z, say x, is larger than 1. Since $y^z = x$, either y or z must be larger than 1. If y is larger than 1, then $z = x^y$ is also larger than 1. If z is larger than 1, then $y = z^z$ is also larger than 1. Hence, if one of the numbers is larger than 1, they all are. But as the product of the numbers equals 1, they cannot all be larger than 1. Hence they are all less than or equal to 1. If one of them is less than one, it now follows that the product is less than 1. Therefore it must be that $x = y = z = 1$.

6. *Let X be a non-empty subset of a finite group G. For $g \in G$, the set Xg is defined by $\{xg | x \in X\}$. Suppose for every pair of elements, $g_1, g_2 \in G$, either $Xg_1 = Xg_2$ or $Xg_1 \cap Xg_2 = \emptyset$. Show $X = Hw$ for some subgroup H of G and some element $w \in G$.*

   Solution: Pick $w \in X$ and let $H = Xw^{-1}$. We show H is a subgroup by showing if $g, h \in H$, then $gh^{-1} \in H$. Let e denote the identity of G. Since $ww^{-1} = e \in H$, $g = ge \in H \cap Hg$. Hence $Xw^{-1} \cap Xw^{-1}g \neq \emptyset$. Therefore, by assumption, $Xw^{-1} = Xw^{-1}g$. Similarly, $Xw^{-1} = Xw^{-1}h$. Thus, $Xw^{-1}g = Xw^{-1}h$, i.e. $Hg = Hh$. Therefore, $Hgh^{-1} = H$ and $egh^{-1} = gh^{-1} \in H$.