Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Bradley University April 3, 2009
Solutions

1. *Let x, y, z be real numbers such that*

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{x+y+z}.$$

Prove that either $x+y=0$ or $y+z=0$ or $z+x=0$.

Solution:

Algebraic solution: By finding a common denominator and clearing fractions, the given condition becomes

$$(x+y+z)(xy+yz+zx)-xyz=0.$$

By expanding this we get

$$(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+2xyz=0.$$

On the other hand,

$$(x+y)(y+z)(z+x)=(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+2xyz.$$

Thus the given condition is equivalent to $(x+y)(y+z)(z+x)=0$.

Solution using polynomials: Now x, y, and z are the roots of

$$P(t)=(t-x)(t-y)(t-z).$$

However,

$$P(t)=(t-x)(t-y)(t-z)=t^3-(x+y+z)t^2+(xy+yz+zx)t-xyz.$$

Letting $a=x+y+z$, $b=xy+yz+zx$, and using the fact that $(x+y+z)(xy+yz+zx)=xyz$, we have

$$P(t)=t^3-at^2+bt-ab.$$

This can be factored as $P(t)=(t^2+b)(t-a)$. Since $P(x)=P(y)=P(z)=0$, either $a=x$ or $a=y$ or $a=z$. This means either $y+z=0$ or $x+z=0$ or $x+y=0$, as required.

2. *There are 1000 bottles of wine, one of which is poisoned. You have 10 rats to which you can give mixtures from the wine bottles. Show that it is possible to prepare 10 mixtures of wine, administer one mixture to each rat* **simultaneously***, and based upon which rats survive, determine which of the bottles is poisoned.*

Solution: Label the bottles in binary from 0 000 000 000 to 1 111 101 000. Thus, there are 10 bits in each label. Make ten mixtures of wine so that the k-th mixture consists of one drop from every bottle whose label has 1 as the k-th bit. Number the rats from 1 through 10 and give the k-th mixture to the k-th rat. Now, write a 10 bit number whose k-th bit is 1 if and only if the k-th rat dies. But the k-th rat dies if and only if there is poison in a bottle whose label has a k-th bit of 1. Therefore, the 10 bit number gives the label on the poisoned bottle.

3. *You have a finite number of lamps, each having a parabolic reflector and thus, each can illuminate a region on the plane which is an infinite parabola and its interior. Prove or disprove:* It is possible to place a finite number of lamps so that each point of the plane is illuminated by at least one lamp.

Solution: It is impossible. We use the following:

Lemma: *If $\mathcal{P}$ is a parabola and $\mathcal{L}$ is a line that is not parallel to the axis of $\mathcal{P}$, then either $\mathcal{P} \cap \mathcal{L}$ consists of two points or $\mathcal{L}$ is tangent to $\mathcal{P}$ or the line does not intersect the parabola.*

Proof: By choosing a suitable coordinate system, we may assume that the parabola has equation $y = x^2$. The axis of this parabola is the line $x = 0$. Assume $\mathcal{L}$ is not parallel to this line. Then $\mathcal{L}$ has equation $y = mx + b$. To find the points of intersection we must find all real solutions to $x^2 = mx + b$. These occur whenever

$$x = \frac{m \pm \sqrt{m^2 + 4b}}{2}.$$

If $m^2 + 4b$ is negative, there are no real solutions. If $m^2 + 4b$ is positive, there are two different solutions. Finally, if $m^2 + 4b$ is zero, there is a unique solution. In this case the line is $y = mx - m^2/4$, which is tangent to $y = x^2$ at the point $(m/2, m^2/4)$.

Solution: Suppose that the plane can be illuminated by a finite number of parabolic lamps. Choose a line that is not parallel to any of the axes of the parabolas. By translating this line, if necessary, we can find a line which is neither parallel to any of the axes of the parabolas nor tangent to any of them. By the lemma, each parabola meets this line in exactly two points. Each parabolic lamp only illuminates the finite segment between the points of intersection of the line and the parabola determined by this lamp. Therefore, a finite number of such lamps cannot illuminate the infinite line, let alone the whole plane.

4. *For x a real number, $\{x\}$ denotes the fractional part of x. For example, $\{5/3\} = 2/3$ and $\{3.14159\} = 0.14159$. Find, with proof, the largest real number x such that*

$$\{5\{4\{3\{2\{x\}\}\}\}\} = x.$$

Solution:

Lemma: *If k is a positive integer and x is a non-negative real number, then $\{k\{x\}\} = \{kx\}$.*

Proof: Let $x = n + \varepsilon$, where n is an non-negative integer and $0 \le \varepsilon < 1$. Then

$$\{kx\} = \{k(n + \varepsilon)\} = \{kn + k\varepsilon\} = \{k\varepsilon\},$$

since kn is an non-negative integer. On the other hand, it is clear that $\{k\{x\}\} = \{k\varepsilon\}$.

Solution: Suppose $\{5\{4\{3\{2\{x\}\}\}\}\} = x$. Since the fractional part of a real number is non-negative, x is non-negative. Therefore, the lemma applies and $\{120x\} = x$. Let x be written in base 120. Since $x = \{120x\}$, x is less than 1 and

$$x = 0.a_1a_2a_3\ldots_{(\text{base } 120)}.$$

Now, $120x = a_1.a_2a_3\ldots_{(\text{base } 120)}$. Thus,

$$\{120x\} = 0.a_2a_3a_4\ldots_{(\text{base } 120)}.$$

Hence, $a_1 = a_2 = a_3 = \cdots$ and all of the "digits" in the base 120 expansion of x are the same. In other words, there is an integer a between 0 and 119, inclusive, such that

$$x = \frac{a}{120} + \frac{a}{120^2} + \frac{a}{120^3} + \cdots.$$

Thus,

$$x = \frac{a}{120}\left(1 + \frac{1}{120} + \left(\frac{1}{120}\right)^2 + \left(\frac{1}{120}\right)^3 + \cdots\right) = \frac{a}{120} \cdot \frac{1}{1 - \frac{1}{120}} = \frac{a}{119}.$$

Since x must be less than 1, the largest a can be is 118. Therefore, the largest solution to the given equation is 118/119.

5. *Seven people come to a party, each with his/her own coat. At the end of the party, the coats are randomly distributed to the people, one coat per person. What is the average number, over all $7!$ possibilities, of the number of people who get their own coat? For example, if there were only two people at the party, there would only be two possibilities — either both people would get back their own coat or each would get the other person's coat. This gives an average of $(2+0)/2 = 1$, in this case.*

   Solution: The average number is 1.

   Solution using the expected value of a random variable: For $i = 1, 2, \ldots, 7$, let x_i be the random variable which is 1 if the i-th person gets back their own coat, 0 otherwise. Since there are 7 coats, the expected value of x_i is $1/7$. Let $X = x_1 + x_2 + \cdots + x_7$. The average number of people who get back their own coat is the expected value of X. Since the expected value of the sum of random variables is the sum of the expected values of the random variables, the expected value of X is $7/7 = 1$.

   Solution by counting: Consider the list of all $5040 = 7!$ possibilities as an array having $7!$ rows and 7 columns. The total number, over all possibilities, of the people who get back their own coat is the number of times that k appears in the k-th column, $k = 1, 2, \ldots 7$. For each k, there are $6!$ rows in which k occurs in the k-th column, corresponding to the $6!$ permutations of the other 6 numbers. Hence, the total, over all k, is $7 \cdot 6!$ or $7!$. Consequently, the average number over all possibilities is $7!/7! = 1$.

6. *Does there exist a power of 5 such that the digits of the number can be rearranged to obtain a* **larger** *power of 5? Justify your answer.*

   Solution: Suppose, by way of contradiction, that such a power, say 5^k, exists. Let 5^m be the larger power of 5 obtained by rearranging the digits. Now, $k < m$. If both numbers have j digits, then

$$10^{j-1} < 5^k < 5^m < 10^j.$$

   Hence $5^{m-k} < 10$ and $m = k+1$. But 5^k and 5^{k+1} are assumed to have the same digits and thus, they are congruent modulo 9. That is, 9 divides $5^{k+1} - 5^k = 4 \cdot 5^k$. This is a contradiction.