

Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Bradley University April 3, 2009

Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Not only must your solutions be correct, statements must be completely justified. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution.

Enjoy the problem solving!

1. Let x, y, z be real numbers such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x + y + z}.$$

Prove that either $x + y = 0$ or $y + z = 0$ or $z + x = 0$.

2. There are 1000 bottles of wine, one of which is poisoned. You have 10 rats to which you can give mixtures from the wine bottles. Show that it is possible to prepare 10 mixtures of wine, administer one mixture to each rat **simultaneously**, and based upon which rats survive, determine which of the bottles is poisoned.
3. You have a finite number of lamps, each having a parabolic reflector and thus, each can illuminate a region on the plane which is an infinite parabola and its interior. Prove or disprove: *It is possible to place a finite number of lamps so that each point of the plane is illuminated by at least one lamp.*
4. For x a real number, $\{x\}$ denotes the fractional part of x . For example, $\{5/3\} = 2/3$ and $\{3.14159\} = 0.14159$. Find, with proof, the largest real number x such that

$$\{5\{4\{3\{2\{x\}\}\}\}\} = x.$$

5. Seven people come to a party, each with his/her own coat. At the end of the party, the coats are randomly distributed to the people, one coat per person. What is the average number, over all $7!$ possibilities, of the number of people who get their own coat? For example, if there were only two people at the party, there would only be two possibilities — either both people would get back their own coat or each would get the other person's coat. This gives an average of $(2 + 0)/2 = 1$, in this case.
6. Does there exist a power of 5 such that the digits of the number can be rearranged to obtain a **larger** power of 5? Justify your answer.