

Student Mathematics Competition
 Illinois Section of the
 Mathematical Association of America
 Eastern Illinois University — April 4, 2008
 Solutions

1. *There are three different numbers, A , B , and C . There are thirty cards on the table, ten have the number A written on them, ten have B on them, and ten have C on them. It is the case that whenever five cards are chosen, there are five other cards such that the total of the numbers on the ten cards is 0. One of the cards has a 2008 on it. What are the numbers on the other cards? Justify your answer.*

Solution: The numbers are -2008 , 0 and 2008 . Let $A < B < C$. The largest sum of the numbers on five cards is $5C$ and the smallest sum is $5A$. Let x be the sum of the five cards that will balance the 5 C cards and y the sum of the five cards which balance the five A cards. Then $x + 5C = 0$ and $x \geq 5A$ so

$$5A + 5C \leq x + 5C = 0.$$

This means $A + C \leq 0$. Similarly, $y + 5A = 0$ and $5C \geq y$ so $5A + 5C \geq y + 5A = 0$. Hence, $A + C \geq 0$. It follows that $A + C = 0$ and thus, A and C are negatives of each other.

The second largest sum is $4C + B$. Let z be the sum of the numbers on the five cards which balance the five cards — four C and one B . If $z = 5A$, then $0 = 4C + B + z = B + A$ and $B = -A = C$, which is impossible since the three numbers are different. Hence z is at least as large as the second smallest sum which means

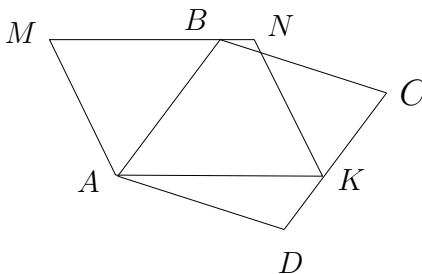
$$4A + B \leq z.$$

In a similar way, if w is the sum of the five cards which balance the five cards — four A and one B , then $w \leq 4C + B$. Hence,

$$0 = (4A + B) + w \leq (4A + B) + (4C + B) \leq z + (4C + B) = 0.$$

Thus, $B = 0$. Therefore, the numbers are -2008 , 0 and 2008 , as claimed.

2. *The vertex B of the parallelogram $ABCD$ belongs to the side MN of the parallelogram $AMNK$ and the vertex K of the parallelogram $AMNK$ belongs to the side CD of the parallelogram $ABCD$, as shown below. Which parallelogram has a larger area, $ABCD$ or $AMNK$? Justify your answer.*



Solution: Draw the line BK . Now, $\triangle ABK$ has a base (AK) equal to the base of parallelogram $AMNK$ and a height, perpendicular to that base, equal to the height of $AMNK$. Therefore, the area of $\triangle ABK$ is one-half the area of parallelogram $AMNK$.

Also, $\triangle ABK$ has a base (AB) equal to the base of parallelogram $ABCD$ and a height, perpendicular to that base, equal to the height of $ABCD$. Therefore, as before, the area of $\triangle ABK$ is equal to one-half the area of parallelogram $ABCD$.

Combining these results gives that the two parallelograms have the same area.

3. Find all triples (x, y, z) of real numbers such that

$$x^3 - y^2 = y^3 - z^2 = z^3 - x^2 = 100.$$

Solution: There is only one triple of numbers which satisfy these equations.

Note that $x^3 = 100 + y^2$, $y^3 = 100 + z^2$, and $z^3 = 100 + x^2$. In particular, all unknowns, x , y , and z are positive.

The given equations imply

$$\begin{aligned} x^3 - y^3 &= y^2 - z^2 \\ y^3 - z^3 &= z^2 - x^2 \end{aligned}$$

Factoring each of these gives

$$\begin{aligned} (x - y)(x^2 + xy + y^2) &= (y - z)(y + z) \\ (y - z)(y^2 + yz + z^2) &= (z - x)(z + x) \end{aligned}$$

If $x > y$, then from the first of these equations we get $y > z$, since all unknowns are positive. Hence, from the second equation $z > x$. Combining these inequalities gives $x > y > z > x$, which is impossible. A similar contradiction arises from the assumption that $x < y$. Therefore, $x = y$. It follows that $x = y = z$.

Therefore, we need to find the positive values of x such that $x^3 = 100 + x^2$ or $x^2(x - 1) = 100$. The positive integral solutions to this equation, if there are any, must be less than 10 and a divisor of 100. By examination, one such value is 5. Since $x^3 - x^2 - 100 = (x - 5)(x^2 + 4x + 20)$ and $x^2 + 4x + 20 = 0$ has no real roots, the only triple of real numbers which satisfies the given system is $(5, 5, 5)$.

4. All of the positive integers are printed, in order, on an infinite strip:

12345678910111213...99100101102...

Then, all zeroes are erased to give:

1234567891 111213...991 1 11 2...

Then all the spaces are removed and the tape is cut into 4-digit strips:

1234	5678	9111	1213	...
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Show that every single 4-digit sequence, $abcd$, with $a, b, c, d \in \{1, 2, \dots, 9\}$, appears on infinitely many strips.

Solution: Let $abcd$ be a 4-digit sequence. The sequence

1abcd2abcd3abcd4abcd

occurs infinitely often since it is the last 20 digits of infinitely many positive integers. Further, none of the digits of these number are erased. When the tape is cut into strips, the cut is either after the first (leftmost) 1, the first a , the first b , or the first c . In these situations, the following 4-digit strips result:

first 1	abcd	...			
first a	bcd2	abcd	...		
first b	cd2a	bcd3	abcd	...	
first c	d2ab	cd3a	bcd4	abcd	...

In any case, the strip \boxed{abcd} occurs once each time the 20-digit sequence, given above, occurs on the tape. Hence, every 4-digit sequence occurs infinitely often.

5. A point (p, q) is chosen at random from the square

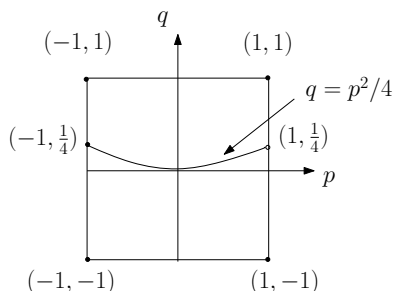
$$S = \{(p, q): -1 \leq p, q \leq 1\}.$$

What is the probability that the quadratic equation $x^2 + px + q = 0$ has two distinct real roots? Justify your answer.

Solution: The quadratic equation $x^2 + px + q = 0$ has two real roots when $p^2 - 4q > 0$ or $q < p^2/4$. Therefore, if $A = \{(p, q): q < p^2/4, -1 \leq p, q \leq 1\}$, then the desired probability is

$$\frac{\text{area}(A)}{\text{area}(S)}.$$

The numerator, $\text{area}(A)$, is area of the region in the square below the parabola, as shown in the following diagram:



This area is

$$\int_{-1}^1 \left(\frac{p^2}{4} - (-1) \right) dp = 2 \int_0^1 \left(\frac{p^2}{4} + 1 \right) dp = 2 \left(\frac{p^3}{12} + p \right) \Big|_0^1 = \frac{13}{6}.$$

Therefore, the probability that the quadratic equation has two real roots is $\frac{13}{24}$.

6. You are given a square and three identical copies of a triangle, $\triangle ABC$, which fits in the square. Suppose that whenever two copies of $\triangle ABC$ are placed in the square, the two copies have a point in common. Show that whenever all three copies of $\triangle ABC$ are placed in the square, there is a point common to all three copies.

Solution: The key observation is the following:

Lemma: Let $\triangle ABC$ be as described in the statement of the problem. Whenever a single copy of $\triangle ABC$ is placed in the square, the center, O , of the square is contained in the triangle.

It follows immediately that no matter how many copies of $\triangle ABC$ are placed in the square, the center will be common to all of them.

Proof of the Lemma: Suppose, by way of contradiction, that $\triangle ABC$ can be placed so that it does not contain the center O . Reflect $\triangle ABC$ in the center of the square to get a new triangle $\triangle A'B'C'$. Since the square is centrally symmetric about O , $\triangle A'B'C'$ is also in the square. Because any two copies of $\triangle ABC$ have a point in common when placed in the square, there is a point X in $\triangle ABC$ and $\triangle A'B'C'$. The image, X' , of the point X , under the central symmetry, is in the images of $\triangle ABC$ and $\triangle A'B'C'$. Since the image of $\triangle A'B'C'$ is $\triangle ABC$, we have that X' is in $\triangle ABC$. Because the line joining two points of a triangle is contained in the triangle, the line XX' is contained in $\triangle ABC$. But O is a point of XX' and thus it must also be in $\triangle ABC$. This contradicts the assumption that O is not in $\triangle ABC$ and the lemma is established.