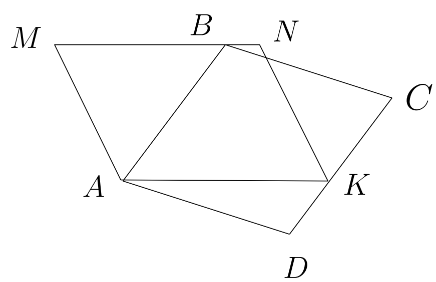
Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Eastern Illinois University April 4, 2008

Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution.

Enjoy the problem solving!

1. There are three different numbers, A, B, and C. There are thirty cards on the table, ten have the number A written on them, ten have B on them, and ten have C on them. It is the case that whenever five cards are chosen, there are five other cards such that the total of the numbers on the ten cards is 0. One of the cards has a 2008 on it. What are the numbers on the other cards? Justify your answer.

2. The vertex B of the parallelogram $ABCD$ belongs to the side MN of the parallelogram $AMNK$ and the vertex K of the parallelogram $AMNK$ belongs to the side CD of the parallelogram $ABCD$, as shown below. Which parallelogram has a larger area, $ABCD$ or $AMNK$? Justify your answer.



3. Find all triples (x, y, z) of real numbers such that

$$x^3 - y^2 = y^3 - z^2 = z^3 - x^2 = 100.$$

4. All of the positive integers are printed, in order, on an infinite strip:

$$12345678910111213 \cdots 99100101102 \cdots$$

Then, all zeroes are erased to give:

$$1234567891\ 111213 \cdots 991\ \ 1\ 11\ 2 \cdots$$

Then all the spaces are removed and the tape is cut into 4-digit strips:

$$\boxed{1234}\ \boxed{5678}\ \boxed{9111}\ \boxed{1213} \cdots$$

Show that every single 4-digit sequence, $abcd$, with $a, b, c, d \in \{1, 2, \ldots 9\}$, appears on infinitely many strips.

5. A point (p, q) is chosen at random from the square

$$\mathcal{S} = \{(p, q)\colon\ -1 \le p, q \le 1\}.$$

What is the probability that the quadratic equation $x^2 + px + q = 0$ has two distinct real roots? Justify your answer.

6. You are given a square and three identical copies of a triangle, $\triangle ABC$, which fits in the square. Suppose that whenever two copies of $\triangle ABC$ are placed in the square, the two copies have a point in common. Show that whenever all three copies of $\triangle ABC$ are placed in the square, there is a point common to all three copies.