Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Western Illinois University — March 30, 2007
Solutions

1. *For x a real number, the greatest integer less than or equal to x is denoted $[x]$. The "fractional part" of x, denoted $\{x\}$ is defined by $\{x\} = x - [x]$. Find all real solutions to the following system:*

$$\begin{aligned}[3x] + \{y\} + x - y &= 1\\ [-y] - \{x\} - x + y &= 1\end{aligned}$$

   Solution: The solution is $x = -1$ and $y = -5$.
   Adding these two equations gives

$$[3x] + [-y] + (\{y\} - \{x\}) = 2.$$

   Thus, $\{y\} - \{x\}$ is an integer. Since $\{x\}$ and $\{y\}$ are less than 1 and at least as large as zero, $-1 < \{y\} - \{x\} < 1$. Hence $\{y\} - \{x\} = 0$. Therefore, $x = a + \varepsilon$ and $y = b + \varepsilon$, where a and b are integers and $0 \le \varepsilon < 1$. Substituting these into the first equation gives

$$[3(a + \varepsilon)] + \varepsilon + a + \varepsilon - b - \varepsilon = 1.$$

   Since $[3(a + \varepsilon)]$, a, and b are integers, it follows that ε is an integer. That is, $\varepsilon = 0$. Thus $x = a$, $y = b$ and, because x and y are integers,

$$\begin{aligned}3a + a - b &= 1\\ -b - a + b &= 1\end{aligned}$$

   Hence, $a = -1 = x$ and $b = -5 = y$.

2. *Let g be the function defined by*

$$g(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0\\ 1, & x = 0\end{cases}$$

   Let h be the function defined by $h(x) = \displaystyle\int_x^{\pi} g(t)\ dt$. Find the area of the region bounded by the curve $y = h(x)$, $x = 0$, $x = \pi$, and the x-axis.

   Solution: The area of this region is 2.

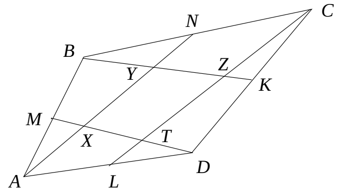
   The required area is the value of the following integral

$$\int_0^{\pi} \left(\int_x^{\pi} g(t)\ dt \right)\ dx.$$

   This integral is taken over the region $\mathcal{R} = \{(t, x)\colon x \le t \le \pi, 0 \le x \le \pi\}$, which is the triangle in the tx-plane bounded by $t = x$, $t = \pi$, and $x = 0$. Therefore, the region $\mathcal{R}$ can also be described as $\{(t, x)\colon 0 \le x \le t, 0 \le t \le \pi\}$. Hence the desired integral is

$$\int_0^{\pi} \left(\int_0^{t} g(t)\ dx \right) dt = \int_0^{\pi} t g(t)\ dt = \int_0^{\pi} \sin t\ dt = -\cos t|_0^{\pi} = 2.$$

3. *Let $ABCD$ be a convex quadrilateral. Let M, N, K, and L be the midpoints of AB, BC, CD, and DA, respectively, as shown below. Finally let X, Y, Z, and T be the intersections of AN with DM, BK with AN, CL with BK, and DM with CL, respectively. If the area of quadrilateral $ABCD$ is 3000, the area of $YNCZ$ is 388, and the area of $AXTL$ is 513, what is the area of $XYZT$? Justify your answer.*



Solution: The area of $XYZT$ is 599.

Draw line AC. Since N is the midpoint of BC, the area of $\triangle ABN$ equals the area of $\triangle ANC$ and each is equal to half of the area of $\triangle ABC$. Similarly, the area of $\triangle ACL$ equals the area of $\triangle CLD$ which equals half of the area of $\triangle ACD$. In particular, the area of $\triangle ANC$ plus the area of $\triangle ACL$ is half of the area of $ABCD$. Thus, the area of $ANCL$ is 1500.

On the other hand, $ANCL$ is composed of $YNCZ$, $XYZT$, and $AXTL$. Therefore, the area of $XYZT$ is 1500 - (388 + 513) = 599.

4. *Let a, b, c be positive real numbers. Prove that $a^2+b^2+c^2 < 2ab+2ac+2bc$ if and only if there exists a triangle with sides of length $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$.*

Solution: Now,

$$\begin{aligned}
2ab+2ac+2bc-a^2-b^2-c^2 &= 4ab-(a^2+b^2+c^2+2ab-2ac-2bc)\\
&= (2\sqrt{ab})^2-(a+b-c)^2\\
&= ((2\sqrt{ab})-(a+b-c))((2\sqrt{ab})+(a+b-c))\\
&= (c-(a-2\sqrt{ab}+b))((a+2\sqrt{ab}+b)-c)\\
&= ((\sqrt{c})^2-(\sqrt{a}-\sqrt{b})^2)((\sqrt{a}+\sqrt{b})^2-(\sqrt{c})^2)\\
&= (\sqrt{c}-\sqrt{a}+\sqrt{b})(\sqrt{c}+\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})\\
&= (-\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}+\sqrt{b}+\sqrt{c})
\end{aligned}$$

If a triangle exists, then each term in this product is positive. Hence, $a^2+b^2+c^2 < 2ab+2ac+2bc$.

If $a^2+b^2+c^2 < 2ab+2ac+2bc$, then this product is positive. Without loss of generality, we may assume that $0 < a \le b \le c$. Surely, $\sqrt{a}+\sqrt{b}+\sqrt{c} > 0$, $\sqrt{a}+(-\sqrt{b}+\sqrt{c}) > 0$, and $(-\sqrt{a}+\sqrt{b})+\sqrt{c} > 0$. Therefore, the fourth factor of the product, $\sqrt{a}+\sqrt{b}-\sqrt{c} > 0$. It follows that a triangle exists whose sides have lengths $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$.

5. *Evaluate the following limit:*

$$\lim_{n\to\infty}\frac{(2^3-1)(3^3-1)(4^3-1)\cdots(n^3-1)}{(2^3+1)(3^3+1)(4^3+1)\cdots(n^3+1)}.$$

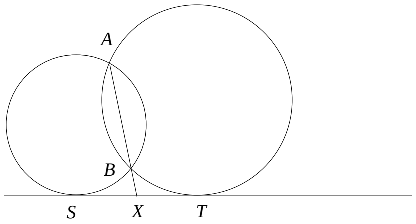
Solution: The value of the limit is $\frac{2}{3}$.

Observe that for every value of k we have that $k^2+k+1=(k+1)^2-(k+1)+1$. Thus,

$$\begin{aligned}\frac{(2^3-1)(3^3-1)(4^3-1)\cdots(n^3-1)}{(2^3+1)(3^3+1)(4^3+1)\cdots(n^3+1)} &= \frac{(2-1)(3-1)\cdots(n-1)(2^2+2+1)(3^2+3+1)\cdots(n^2+n+1)}{(2+1)(3+1)\cdots(n+1)(2^2-2+1)(3^2-3+1)\cdots(n^2-n+1)}\\ &= \frac{1\cdot 2\cdots(n-1)}{3\cdot 4\cdots(n+1)}\cdot\frac{n^2+n+1}{2^2-2+1}\\ &= \frac{2}{n(n+1)}\cdot\frac{n^2+n+1}{3}=\frac{2n^2+2n+2}{3n^2+3n}.\end{aligned}$$

Thus, the desired limit is $\lim_{n\to\infty}\dfrac{2n^2+2n+2}{3n^2+3n}=\dfrac{2}{3}$.

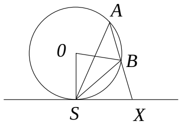
6. *Let ST be a common tangent to two circles (of different sizes) which meet at A and B. Let X be the intersection of lines AB and ST, as shown. Prove that $SX = XT$.*



(You may use without proof that if R, S, T are three points on a circle with center O, then $\angle RST = \frac{1}{2}\angle ROT$.)

Solution: We will prove that the product $AX \cdot BX$ equals both SX^2 and XT^2, from which the result follows.

Let O be the center of the circle containing A, B, and S, as shown.



Since $OS \perp SX$, $\angle OSB + \angle BSX = \frac{\pi}{2}$. Further, $\triangle BOS$ is an isosceles triangle with $OB = OS$. Hence, $\angle OSB = \frac{1}{2}(\pi - \angle BOS)$. On the other hand, since A, B, S are three points on the circle, $\angle BAS = \frac{1}{2}\angle BOS$. Therefore,

$$\angle BSX = \frac{\pi}{2} - \angle OSB = \frac{1}{2}\angle BOS = \angle BAS.$$

Since $\angle BSX = \angle BAS = \angle SAX$ and $\angle SXB = \angle AXS$, $\triangle BSX$ and $\triangle SAX$ are similar. Hence,

$$\frac{SX}{BX} = \frac{AX}{SX}.$$

That is, $AX \cdot BX = SX^2$. Similarly, $AX \cdot BX = XT^2$. Thus, $SX = XT$, as desired.