Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Western Illinois University March 30, 2007

Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution.

Enjoy the problem solving!

1. For x a real number, the greatest integer less than or equal to x is denoted $[x]$. The "fractional part" of x, denoted $\{x\}$ is defined by $\{x\} = x - [x]$. Find all real solutions to the following system:

$$\begin{aligned} [3x] + \{y\} + x - y &= 1 \\ [-y] - \{x\} - x + y &= 1 \end{aligned}$$

2. Let g be the function defined by

$$g(x) = \begin{cases} \dfrac{\sin(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Let h be the function defined by $h(x) = \displaystyle\int_x^{\pi} g(t)\, dt$. Find the area of the region bounded by the curve $y = h(x)$, $x = 0$, $x = \pi$, and the x-axis.

3. Let $ABCD$ be a convex quadrilateral. Let M, N, K, and L be the midpoints of AB, BC, CD, and DA, respectively, as shown below. Finally let X, Y, Z, and T be the intersections of AN with DM, BK with AN, CL with BK, and DM with CL, respectively. If the area of quadrilateral $ABCD$ is 3000, the area of $YNCZ$ is 388, and the area of $AXTL$ is 513, what is the area of $XYZT$? Justify your answer.

4. Let a, b, c be positive real numbers. Prove that $a^2 + b^2 + c^2 < 2ab + 2ac + 2bc$ if and only if there exists a triangle with sides of length $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$.

5. Evaluate the following limit:

$$\lim_{n \to \infty} \frac{(2^3 - 1)(3^3 - 1)(4^3 - 1) \cdots (n^3 - 1)}{(2^3 + 1)(3^3 + 1)(4^3 + 1) \cdots (n^3 + 1)}.$$

6. Let ST be a common tangent to two circles (of different sizes) which meet at A and B. Let X be the intersection of lines AB and ST, as shown. Prove that $SX = XT$.

(You may use without proof that if R, S, T are three points on a circle with center O, then $\angle RST = \frac{1}{2}\angle ROT$.)