# Student Mathematics Competition
# Illinois Section of the
# Mathematical Association of America
# North Central College April 7, 2006
# Solutions

1. *There are several people sitting at a round table. Each is wearing a cap, which is either red, white, blue, or green. Each person can see all of the caps except her or his own. Some of the people make the following true statements:*

   **Andy:** *I see exactly 6 red and 11 blue caps.*

   **Betty:** *I see exactly 12 blue and 8 green caps.*

   **Cathy:** *I see exactly 7 green and 10 white caps.*

   **Donald:** *The number of red caps that I see is more than the number of red caps that Betty sees.*

   *It is known that no two of Andy, Betty, Cathy, and Donald have a cap of the same color.*

   (a) *Find the color of the cap worn by each of Andy, Betty, Cathy, and Donald.*

   (b) *Find the number of people in the room.*

   (c) *Find the number of caps of each color.*

   **Solution:** A person sees all caps but his/her own. Donald sees more red caps than Betty, so Betty must wear a red cap. Betty sees more blue caps than Andy and more green caps than Cathy, thus Andy is wearing a blue cap and Cathy is wearing a green cap. Since all four have caps of different color, Donald must be wearing a white cap.
   Thus, there are 12 blue caps and 8 green caps — Betty sees all of them. Cathy sees all 10 white caps and Andy sees all 6 red caps. Therefore, there are $12 + 8 + 10 + 6 = 36$ people in the room.

2. *Let $A_1, A_2, \ldots, A_n$ be $n$ points in the plane such that each triangle*

   $$\triangle A_i A_j A_k,\ \textit{for } i, j, k \in \{1, 2, \ldots, n\}, i \neq j \neq k \neq i$$

   *is obtuse. Show that one can find one more point $A_{n+1}$ such that each triangle*

   $$\triangle A_i A_j A_k,\ \textit{for } i, j, k \in \{1, 2, \ldots, n+1\}, i \neq j \neq k \neq i$$

   *is obtuse.*

   **Solution:** Let $A_1, A_2, \ldots, A_n$ satisfy the hypothesis of the problem. For $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$, consider the infinite strip whose boundaries are the two lines which are perpendicular to the line segment $A_iA_j$ and which pass through $A_i$ or $A_j$. Note that $\triangle A_iXA_j$ is obtuse provided $X$ is point outside this strip. There are $\binom{n}{2}$ such strips. Let $w$ be the maximum width of all of these strips.

   The strips do not cover the entire plane. In fact, if a disc of radius $R$ contains all of the points, then the intersection of one of the infinite strips and the disc has area at most $2Rw$, so the area of all of the strips is at most $2Rw\binom{n}{2}$. By taking $R$ sufficiently large, we see that there is a point inside a circle of radius $R$ which is in none of the strips. Let $A_{n+1}$ be such a point. By choice, each triangle $A_iA_jA_{n+1}$, for $i \neq j$, $i, j \in \{1, 2, \ldots n\}$ is obtuse. It is given that the triangles

   $$\triangle A_i A_j A_k,\ \text{for } i, j, k \in \{1, 2, \ldots, n\}, i \neq j \neq k \neq i$$

   are obtuse. Consequently, all the triangles from the new set are obtuse.

3. *Solve the following equation for $x$ a real number:*

$$\sqrt{\frac{x-2}{2006}}+\sqrt{\frac{x-3}{2005}}+\sqrt{\frac{x-4}{2004}}+\sqrt{\frac{x-5}{2003}}=\sqrt{\frac{x-2006}{2}}+\sqrt{\frac{x-2005}{3}}+\sqrt{\frac{x-2004}{4}}+\sqrt{\frac{x-2003}{5}}.$$

**Solution:** The only solution is 2008. Transposing all terms to one side gives the equivalent equation

$$\begin{aligned}0=\left(\sqrt{\frac{x-2006}{2}}-\sqrt{\frac{x-2}{2006}}\right) &+ \left(\sqrt{\frac{x-2005}{3}}-\sqrt{\frac{x-3}{2005}}\right)\\ &+ \left(\sqrt{\frac{x-2004}{4}}-\sqrt{\frac{x-4}{2004}}\right)+\left(\sqrt{\frac{x-2003}{5}}-\sqrt{\frac{x-5}{2003}}\right). \qquad (*)\end{aligned}$$

Consider $\sqrt{\frac{x-a}{b}}-\sqrt{\frac{x-b}{a}}$, $a>b$. By rationalizing,

$$\sqrt{\frac{x-a}{b}}-\sqrt{\frac{x-b}{a}}=\frac{\frac{x-a}{b}-\frac{x-b}{a}}{\sqrt{\frac{x-a}{b}}+\sqrt{\frac{x-b}{a}}}=\frac{(a-b)(x-(a+b))}{a\sqrt{b(x-a)}+b\sqrt{a(x-b)}}.$$

This expression is zero when $x=a+b$, is positive when $x>a+b$, and is negative when $x<a+b$. Thus the right hand side of equation (*) is the sum of four terms each of which is zero when $x=2008$, is positive when $x>2008$, and is negative when $x<2008$. Hence the only solution to the original equation is $x=2008$.

4. *What are the dimensions of the rectangular parallelepiped of maximum volume subject to the condition that the sum of the areas of five of the faces is 120 square feet?*

**Solution:** Let the sides of the parallelepiped be $x$, $y$, $z$. We want to find the maximum value of $xyz=V$ subject to the condition that $2xy+2xz+yz=120$. By the arithmetic-geometric mean

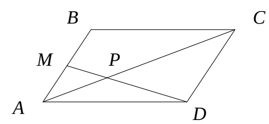
$$\sqrt[3]{(2xy)(2xz)(yz)}\leq\frac{2xy+2xz+yz}{3},$$

with equality if and only if $2xy=2xz=yz$.

Thus, the maximum volume occurs when $2xy=2xz=yz$ or $y=z=2x$. Since $2xy+2xz+yz=120$, we get that the maximum volume occurs when $x=\sqrt{10}$, $y=z=2\sqrt{10}$.

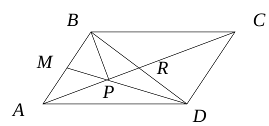
Notice that if two of the parallelepipeds of maximum volume are placed so that their sides of area $yz$ are together, the resulting solid is a cube.

5. *Suppose $ABCD$ is a parallelogram, $M$ is the midpoint of $AB$, and $P$ is the intersection of $MD$ and $AC$, as shown below. Find the ratio of the area of $\triangle AMP$ to the area of parallelogram $ABCD$.*




**Solution:** The ratio of the area of $\triangle AMD$ to the area of $ABCD$ is $:12$. Observe that a line is drawn from a vertex of a triangle to the midpoint of the opposite side divides the triangle into two triangles with the same base and the same height. Therefore, the area of each of the smaller triangles is half the area of the large triangle.

Let $R$ be the intersection of $BD$ and $AC$. Construct a line from $P$ to $B$, as shown below,




Since $R$ is the intersection of the two diagonals of a parallelogram, $R$ is the midpoint of $BD$. Thus, considering $\triangle PBD$ and the line from $P$ to the midpoint, $R$, of $BD$, we see that $\triangle PRD$ and $\triangle PRB$ have equal areas. From $\triangle ABD$ and the line joining $A$ to the midpoint, $R$, of $BD$, we have that $\triangle ARD$ and $\triangle ARB$ have the same area. Since $\triangle APB$ is the difference of $\triangle ARB$ and $\triangle PRB$ and $\triangle APD$ is the difference of $\triangle ARD$ and $\triangle PRB$, it follows that $\triangle APB$ and $\triangle APD$ have equal areas. By considering $\triangle APB$ and the midpoint $M$ of $AB$, it is possible to conclude that $\triangle AMP$ has an area which is half that of $\triangle APB$.

The area of $ABCD$ is twice the area of $\triangle ABD$ which is twice the area of $\triangle AMD$. However, the area of $\triangle AMD$ is the sum of the areas of $\triangle AMP$ and $\triangle APD$ (whose area is twice the area of $\triangle AMP$). Therefore, the area of $\triangle AMD$ is three times the area of $\triangle AMP$, and the area of $ABCD$ is twelve times the area of $\triangle AMP$.

6. *Let $p$ and $q$ be real numbers with $p > q$. Compute*

$$\int_0^1 \left(1 - x^{1/p}\right)^q \, dx - \int_0^1 \left(1 - x^{1/q}\right)^p \, dx.$$

**Solution:** The difference is zero! The curve $y = (1 - x^{1/p})^q$ passes through the points $(0, 1)$ and $(1, 0)$. Also, the first integral is the area in the first quadrant between this curve and the coordinate axes. Solving for $x$ gives $x = (1 - y^{1/q})^p$. Hence, the area between this curve and the coordinate axes (using a "$y$" integration) is $\int_0^1 (1 - y^{1/q})^p \, dy$. Since the area is the same, in either case,

$$\int_0^1 \left(1 - x^{1/p}\right)^q \, dx - \int_0^1 \left(1 - y^{1/q}\right)^p \, dy = 0.$$