Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
North Central College April 7, 2006

Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution.

Enjoy the problem solving!

1. There are several people sitting at a round table. Each is wearing a cap, which is either red, white, blue, or green. Each person can see all of the caps except her or his own. Some of the people make the following true statements:

   Andy: I see exactly 6 red and 11 blue caps.

   Betty: I see exactly 12 blue and 8 green caps.

   Cathy: I see exactly 7 green and 10 white caps.

   Donald: The number of red caps that I see is more than the number of red caps that Betty sees.

   It is known that no two of Andy, Betty, Cathy, and Donald have a cap of the same color.

   (a) Find the color of the cap worn by each of Andy, Betty, Cathy, and Donald.

   (b) Find the number of people in the room.

   (c) Find the number of caps of each color.

2. Let $A_1, A_2, \ldots, A_n$ be n points in the plane such that each triangle

   $$\triangle A_i A_j A_k, \text{ for } i, j, k \in \{1, 2, \ldots, n\}, i \neq j \neq k \neq i$$

   is obtuse. Show that one can find one more point A_{n+1} such that each triangle

   $$\triangle A_i A_j A_k, \text{ for } i, j, k \in \{1, 2, \ldots, n+1\}, i \neq j \neq k \neq i$$

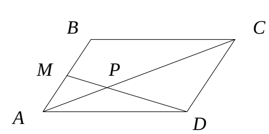
   is obtuse.

3. Solve the following equation for x a real number:

   $$\sqrt{\frac{x-2}{2006}} + \sqrt{\frac{x-3}{2005}} + \sqrt{\frac{x-4}{2004}} + \sqrt{\frac{x-5}{2003}} = \sqrt{\frac{x-2006}{2}} + \sqrt{\frac{x-2005}{3}} + \sqrt{\frac{x-2004}{4}} + \sqrt{\frac{x-2003}{5}}.$$

4. What are the dimensions of the rectangular parallelepiped of maximum volume subject to the condition that the sum of the areas of five of the faces is 120 square feet?

5. Suppose $ABCD$ is a parallelogram, M is the midpoint of AB, and P is the intersection of MD and AC, as shown below. Find the ratio of the area of $\triangle AMP$ to the area of parallelogram $ABCD$.



6. Let p and q be real numbers with $p > q$. Compute

   $$\int_0^1 \left(1 - x^{1/p}\right)^q \, dx - \int_0^1 \left(1 - x^{1/q}\right)^p \, dx.$$