Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Knox College April 8, 2005
Solutions

1. *Find all real numbers that satisfy the following equation:*

$$(|x-1|+|x+1|)(2x^2-x^4-3)=-4.$$

Solution: The equation can be rewritten as

$$(|x-1|+|x+1|)((x^2-1)^2+2)=4.$$

Now,

$$|x+1|+|x-1|=\begin{cases} 2x, & 1<x \\ 2, & -1\leq x\leq 1 \\ -2x, & x<-1 \end{cases}$$

In particular, $|x-1|+|x+1|\geq 2$, for all real x. Clearly, $(x^2-1)+2\geq 2$, for all real x. Thus, if x satisfies the given equation,

$$4=(|x-1|+|x+1|)((x^2-1)^2+2)\geq 4.$$

Therefore, $|x-1|+|x+1|=2$ and $(x^2-1)^2+2=2$. This means $x=1$ or $x=-1$. Both $x=1$ and $x=-1$ satisfy the equation.

2. *Let S be a set of real numbers such that if x and y are in S, then $x+y\in S$, $x-y\in S$, and $\sqrt{x^2+y^2}\in S$. Suppose $\sqrt{2}+\sqrt{3}\in S$. Show that $\sqrt{37}+\sqrt{73}\in S$.*

Solution: By assumption, if $x\in S$ and $x>0$, then $\sqrt{x^2+x^2}=x\sqrt{2}\in S$. Since

$$x\sqrt{n+1}=\sqrt{(x\sqrt{n})^2+x^2},$$

it follows by induction that if $x\in S$, $x>0$, and n is a positive integer, then $x\sqrt{n}\in S$.
If $\sqrt{2}+\sqrt{3}\in S$, then $(\sqrt{2}+\sqrt{3})\sqrt{2}$ and $(\sqrt{2}+\sqrt{3})\sqrt{3}$ are both in S. Thus,

$$(\sqrt{2}+\sqrt{3})\sqrt{3}-(\sqrt{2}+\sqrt{3})\sqrt{2}=\sqrt{6}+3-2-\sqrt{6}=1\in S.$$

Therefore, $\sqrt{n}\in S$ for every positive integer n. In particular, $\sqrt{37}$ and $\sqrt{73}$ are in S. The result follows immediately.

3. *Determine all values of n for which the integers $1,2,\ldots,n$ can be arranged as*

$$a_1,a_2,\ldots a_n$$

so that for $1\leq i<j\leq n$, the arithmetic mean of a_i and a_j is not one of $a_{i+1},\ldots,a_{j-1}$.

Solution: It is possible to find arrangements for every positive integer n.
Suppose $a_1,a_2,\ldots,a_n$ is an arrangement which satisfies the condition for n. Consider the arrangement $b_1,b_2,\ldots,b_{2n}$, where

$$b_j=\begin{cases} 2a_j, & j=1,2,\ldots,n \\ 2a_{j-n}-1, & j=n+1,n+2,\ldots,2n \end{cases}$$

Then, $b_1, b_2, \ldots, b_n$ is an arrangement of $2, 4, \ldots, 2n$ and $b_{n+1}, b_{n+2}, \ldots, b_{2n}$ is an arrangement of $1, 3, \ldots, 2n-1$. If $1 \leq i \leq n < j \leq 2n$, then b_i is even while b_j is odd. Therefore, the arithmetic mean of b_i and b_j is not one of the integers $b_{i+1}, \ldots, b_{j-1}$.

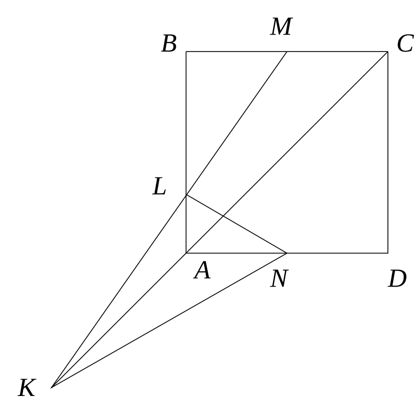
Suppose $1 \leq i < j \leq n$. Let m be the arithmetic mean of b_i and b_j. Hence, $m = (b_i + b_j)/2 = a_i + a_j$. If $m = b_k$, where $i < k < j$, then $m = 2a_k$. Therefore, a_k is the arithmetic mean of a_i and a_j. This is impossible. Hence the arithmetic mean of b_i and b_j is not one of $b_{i+1}, \ldots, b_{j-1}$.

If $n \leq i < j \leq 2n$, a similar argument shows that the arithmetic mean of b_i and b_j is not one of $b_{i+1}, \ldots, b_{j-1}$.

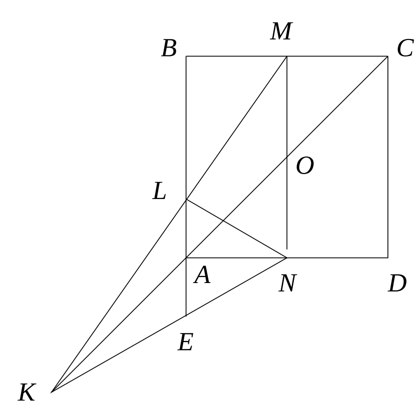
Since the arrangement 1 satisfies the condition for $n = 1$, it follows from the previous paragraphs that the condition is satisfied for $n = 2^k$, where k is a non-negative integer.

To complete the proof, observe that if $c_1, c_2, \ldots, c_m$ is an arrangement which satisfies the condition for some value m, then deleting m from this arrangement gives an arrangement of $1, 2, \ldots m-1$ with the required property.

4. *Let $ABCD$ be a square, M the midpoint of BC, and N the midpoint of AD, as shown below. Further, let K be any point on the diagonal AC, extended in the direction from C to A. Let L be the intersection point of KM and AB. Show that $\angle LNA = \angle KNA$.*



Solution: Connect M to N, then the intersection of MN and the diagonal AC is O, the center of the square. Let E be the intersection of BA, extended through vertex A, and KN, as shown. Clearly, $MO = ON = \frac{1}{2}AB$, so KO is the median of $\triangle KMN$.



Because $LE \parallel MN$, $\triangle KLE$ is similar to $\triangle KMN$. Hence, KA is the median of $\triangle KLE$. Therefore, $LA = AE$, by the definition of the median.

Since $ABCD$ is a square, $\triangle LAN$ and $\triangle EAN$ are right triangles. These triangles share a common side, AN, and, by the previous paragraph, $LA = AE$. Hence, $\triangle LAN$ is congruent to $\triangle EAN$ by SAS and, consequently, $\angle LNA = \angle ANE$, as needed.

5. *The positive integers are written in decimal notation and in order on an infinite strip. Then, all zeroes are deleted. The digits are shifted to the left to eliminate any spaces. The result begins*

$$123456789111121314151617181922122\cdots$$

This strip can be viewed as a strip containing the terms of a sequence of ten-digit integers, where the terms are written next to each other. The first few terms of this sequence are

$$1234567891, 1112131415, 1617181922, \ldots$$

Show that every ten-digit number which does not contain a zero must be a term of this sequence.

Solution: Let $a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}$ be an arbitrary ten-digit number. Let B denote this number. Consider the 109-digit number

$$B1B1B1B1B1B1B1B1B1B$$

which is on the strip before zeroes are deleted. No digits of this number are deleted in forming the second sequence of digits. One of the digits of the first ten digits of this 109-digit number must be the first digit of a term of the sequence of ten-digit numbers. If that digit is a_1, the sequence contains B. If that digit is a_2, the sequence contains the ten-digit number $a_2a_3a_4a_5a_6a_7a_8a_9a_{10}1$ followed by B. If that digit is a_3, the sequence contains $a_3a_4a_5a_6a_7a_8a_9a_{10}1a_1$, $a_2a_3a_4a_5a_6a_7a_8a_9a_{10}1$, and B. Continuing in this manner, we see that, in any case, the sequence contains B.

6. *Let p be a prime. Suppose the period of the repeating decimal $\frac{1}{p}$ has length $2k$, k an integer. Let $\frac{1}{p} = 0.\overline{a_1a_2\ldots a_{2k}}$. Prove that the average of the digits $a_1, a_2, \ldots, a_{2k}$ is 4.5.*

Solution: Because $\frac{10^{2k}}{p} = a_1a_2\ldots a_{2k}.\overline{a_1a_2\ldots a_{2k}}$, $\frac{10^{2k}-1}{p}$ is an integer and thus $p|(10^{2k}-1)$. Therefore, $p|(10^k-1)(10^k+1)$ and, because p is a prime, either $p|10^k-1$ or $p|10^k+1$.
If $p|10^k-1$, then

$$\frac{10^k}{p} - \frac{1}{p} = a_1a_2\cdots a_k.\overline{a_{k+1}a_{k+2}\cdots a_{2k}a_1a_2\cdots a_k} - 0.\overline{a_1a_2\ldots a_{2k}a_{k+1}a_{k+2}\cdots a_{2k}}$$

is an integer. This force $a_{k+1} = a_1$, $a_{k+2} = a_2$, $\ldots a_{2k} = a_k$. Therefore, $\frac{1}{p} = 0.\overline{a_1a_2\cdots a_k}$, a contradiction to the fact that decimal expansion of $\frac{1}{p}$ has length $2k$, not k.

Hence, $p|10^k+1$. Therefore,

$$\frac{10^k}{p} + \frac{1}{p} = a_1a_2\cdots a_k.\overline{a_{k+1}a_{k+2}\cdots a_{2k}a_1a_2\cdots a_k} + 0.\overline{a_1a_2\ldots a_ka_{k+1}a_{k+1}\cdots a_{2k}}$$

is an integer. This integer must be $a_1a_2\cdots a_k.\overline{9}$ or, equivalently, $a_1a_2\cdots a_k+1$. Thus, $a_{k+1}+a_1 = 9$, $a_{k+2}+a_2 = 9$, $\ldots$, $a_{2k}+a_k = 9$. Therefore, $a_1+a_2+\cdots a_{2k} = 9k$ and the arithmetic mean of the $2k$ digits $a_1, a_2, \ldots, a_{2k}$ is 4.5.