Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Knox College April 8, 2005

Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution.

Enjoy the problem solving!

1. Find all real numbers that satisfy the following equation:

$$(|x-1|+|x+1|)(2x^2-x^4-3)=-4.$$

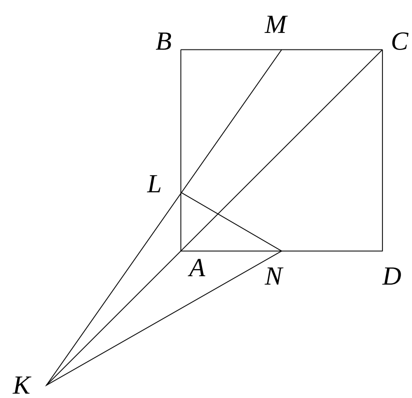
2. Let S be a set of real numbers such that if x and y are in S, then $x+y \in S$, $x-y \in S$, and $\sqrt{x^2+y^2} \in S$. Suppose $\sqrt{2}+\sqrt{3} \in S$. Show that $\sqrt{37}+\sqrt{73} \in S$.

3. Determine all values of n for which the integers $1, 2, \ldots, n$ can be arranged as

$$a_1, a_2, \ldots a_n$$

so that for $1 \leq i < j \leq n$, the arithmetic mean of a_i and a_j is not one of $a_{i+1}, \ldots, a_{j-1}$.

4. Let $ABCD$ be a square, M the midpoint of BC, and N the midpoint of AD, as shown below. Further, let K be any point on the diagonal AC, extended in the direction from C to A. Let L be the intersection point of KM and AB. Show that $\angle LNA = \angle KNA$.



5. The positive integers are written in decimal notation and in order on an infinite strip. Then, all zeroes are deleted. The digits are shifted to the left to eliminate any spaces. The result begins

$$1234567891111213141516171819221222 \cdots$$

This strip can be viewed as a strip containing the terms of a sequence of ten-digit integers, where the terms are written next to each other. The first few terms of this sequence are

$$1234567891, 1112131415, 1617181922, \ldots$$

Show that every ten-digit number which does not contain a zero must be a term of this sequence.

6. Let p be a prime. Suppose the period of the repeating decimal $\frac{1}{p}$ has length $2k$, k an integer. Let $\frac{1}{p} = 0.\overline{a_1 a_2 \ldots a_{2k}}$. Prove that the average of the digits $a_1, a_2, \ldots, a_{2k}$ is 4.5.