Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Roosevelt University - Schaumburg Campus April 2, 2004
Solutions

1. *There are an odd number of terms in a certain arithmetic sequence of integers. The sum of the terms of the sequence is a power of a prime p. Prove that the number of terms of the sequence is also a power of p.*

   Solution: Let n be the number of terms. If $n = 1$, the result is trivially true. Assume that $n > 1$. Let a_1 be the first term and a_n be the last term. Then, the sum of the terms of the sequence is $\frac{a_1 + a_n}{2}n$. Since this is an integer $2|(a_1+a_n)n$. However, n is odd, so $2|(a_1+a_n)$. Therefore, $\frac{a_1 + a_n}{2}$ is an integer and thus n divides the sum of the terms of the sequence. It follows that n is a power of p.

2. *Find positive integers $a_1, a_2, \dots, a_n$ such that*

   $$a_1 + a_2 + \cdots a_n = 100$$

   and such that the product $a_1 a_2 \cdots a_n$ is maximal over all values of n and all choices of positive integers $a_1, a_2, \dots a_n$.

   Solution: Suppose that $a_1, a_2, \dots, a_n$ is some sequence of positive terms whose sum is 100.

   If $a_j \geq 4$ for some j, then

   $$(a_j - 2)2 = 2a_j - 4 \geq 4.$$

   If a_j is be replaced by $a_j - 2$ and 2, the sum of the terms remains the same, but the product is at least as large.

   The maximal product does not contain a factor of 1. For, if $a_i = 1$ and a_j is any other term, the product can be increased by replacing a_j by $a_j + 1$ and deleting a_i. This leaves the sum unchanged.

   Therefore, the maximal product consists of r copies of 2 and s copies of 3. Since the product of three copies of 2 is less than the product of two copies of 3, the maximal product contains at most two copies of 2.

   Neither 100 or 98 is a multiple of 3. Therefore, the maximal product cannot contain zero or one copy of 2. However, 96 is a multiple of three. Thus the maximal product is $2^2 \cdot 3^{32}$.

3. *Find all real solutions to the following equation:*

   $$\frac{x(x+1)}{\sqrt{x-1}} = 6x - 2 - x^2.$$

   Solution: The left hand side of this equation can be rewritten as follows:

   $$\frac{x(x+1)}{\sqrt{x-1}} = \left(\frac{x-1+1}{\sqrt{x-1}}\right)(x+1) = \left(\frac{1}{\sqrt{x-1}} + \sqrt{x-1}\right)(x+1).$$

   By the arithmetic-geometric mean inequality,

   $$\frac{1}{\sqrt{x-1}} + \sqrt{x-1} \geq 2\sqrt{\frac{1}{\sqrt{x-1}}\sqrt{x-1}} = 2.$$

Thus, the left hand side of the given equation is at least $2(x+1)$.

However,

$$6x - 2 - x^2 = 2(x+1) - (x-2)^2.$$

Therefore, the right hand side of the given equation is at most $2(x+1)$. It follows that the only solutions occurs when both sides are equal to $2(x+1)$, i.e. when $x = 2$.

4. *Ross selected five different integers and formed the ten pairwise sums. He then wrote down his results, some of which were wrong, as*

$$2, 4, 5, 7, 7, 8, 10, 11, 12, 13.$$

Chris, who knew the numbers, also computed the pairwise sums and hers were all correct. Chris told Ross that the first three and the last three sums in his list were correct, but some of the others were wrong. What are the five integers? Added in proof: The three smallest correct sums are 2, 4, and 5. The three largest correct sums are 11, 12, and 13.

Solution: Assume the numbers are a, b, c, d, e and that

$$a < b < c < d < e.$$

It follows, easily, that

$$a + b < a + c < b + c < b + d < c + d < c + e < d + e$$

and

$$a + c < a + d < a + e < b + e < c + e.$$

Thus, the smallest pairwise sum is $a + b$, the second smallest is $a + c$, the third is either $b + c$ or $a + d$, the eighth is $c + d$ or $b + e$, the ninth is $c + e$, and the largest is $d + e$.

From the given information we then have

$$\begin{aligned} a + b &= 2 \\ a + c &= 4 \\ c + e &= 12 \\ d + e &= 13 \end{aligned}$$

Either $b + c = 5$ or $a + d = 5$. Since $2a + b + c = 6$, $b + c$ must be even. Therefore, $a + d = 5$. Either $b + e = 11$ or $c + d = 11$. If $b + e = 11$, then

$$16 = (a + d) + (b + e) = (a + b) + (d + e) = 15,$$

a contradiction. Therefore $c + d = 11$.

Consider the equations

$$\begin{aligned} c + d &= 11 \\ c + e &= 12 \\ d + e &= 13 \end{aligned}$$

Adding these equations gives that $c + d + e = 18$. It is now easy to find the numbers as

$$a = -1, b = 3, c = 5, d = 6, e = 7.$$

5. *Suppose x, y, and z are the lengths of the sides of an acute triangle, no two of which are equal. Show there is a positive integer, n, such that x^n, y^n, and z^n are the sides of a triangle that is not acute.*

   Solution: Recall that if a, b, c are the sides of a triangle, then

   $$c^2 = a^2 + b^2 - 2ab\cos\alpha,$$

   where α is the angle between the sides of lengths a and b.

   Without loss of generality, assume that $x < y < z$. Since the triangle is acute, the cosine of the angle between the sides of lengths x and y is positive. Therefore,

   $$z^2 < x^2 + y^2.$$

   Since z^2 is less than the sum of x^2 and y^2, there is a triangle whose sides have lengths x^2, y^2, z^2.

   If this triangle is not acute, we are done. If it is acute, the angle between the sides of this triangle which have lengths x^2 and y^2 is acute. Thus

   $$(z^2)^2 < (x^2)^2 + (y^2)^2.$$

   Since z^4 is less the sum of x^4 and y^4, there is a triangle with sides x^4, y^4, z^4.

   By repeating the argument of the previous paragraph, we see that either there is an integer k such that there is triangle with sides x^{2^k}, y^{2^k}, and z^{2^k} which is not acute or

   $$z^{2^k} < x^{2^k} + y^{2^k}, \text{ for all } k.$$

   If the last case occurs, dividing through by z^{2^k} gives

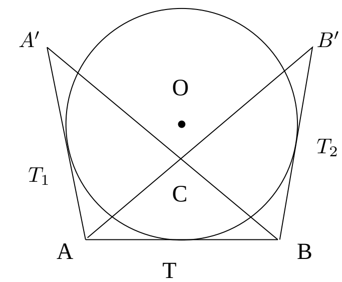
   $$1 < (x/z)^{2^k} + (y/z)^{2^k}, \text{ for all } k.$$

   This is impossible since $x/z < 1$ and $y/z < 1$ and, thus,

   $$\lim_{k\to\infty} (x/z)^{2^k} = \lim_{k\to\infty} (y/z)^{2^k} = 0.$$

   Note: If you begin with $x = \sqrt{3}$, $y = \sqrt{4}$, $z = \sqrt{5}$, you find that there is an acute triangle with sides x, y, z; a right triangle with sides x^2, y^2, z^2; and if $n \geq 3$, then $z^n > x^n + y^n$ so there is no triangle in this case. Thus, it possible that the initial triangle is obtuse, but there is no consequent triangle which is acute.

6. *Suppose $A'A$, AB, and BB' are line segments which are tangent to a circle with center O at points T_1, T_2 and T, as shown below: Let C be the intersection of $A'B$ and AB'. Show that if $A'T_1 = BT$ and $B'T_2 = AT$, then C, O, and T are collinear.*



Solution: By symmetry, $AT_1 = AT$ and $A'T_1 = BT$. Thus

$$A'A = A'T_1 + T_1A = AT + TB$$

and $\triangle A'AB$ is isosceles. Again by symmetry, OA is perpendicular to $A'B$. Thus, BA' is perpendicular to OA and BA' contains the altitude from vertex B to side OA in $\triangle BOA$.

Similarly, AB' contains the altitude from vertex A to side OB in $\triangle BOA$.

Thus C is the point where two of the altitudes of $\triangle BOA$ meet. Since the three altitudes of a triangle intersect and since OT is the altitude from vertex O to side AB in $\triangle BOA$, C is on OT, as required.