Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Illinois College March 28, 2003
Solutions

1. *Suppose x and n are positive integers and that 2^n divides $x^{2003}+1$. Find, in terms of n, the smallest possible value of x.*

   Solution: Note that

   $$x^{2003}+1=(x+1)(x^{2002}-x^{2001}+\cdots+x^2-x+1).$$

   There are 2003 terms in the sum $x^{2002}-x^{2001}+\cdots+x^2-x+1$. This sum is even. Indeed, if x is odd, this sum is the sum of an odd number of odd numbers and if x is even, this sum is one more than a sum of even numbers.

   Therefore, if 2^n divides $x^{2003}+1$, 2^n must divide $x+1$. The smallest positive value for $x+1$ is thus 2^n and the smallest possible value for x is 2^n-1.

2. *At one time, school desks were made so that two students sat at each desk. For a class of 30 students and 15 desks, the teacher wanted to determine k different seating arrangements so that no pair of students sat together in more than one of the k seating arrangements. What is the largest possible value of k for which this can be done? Prove your answer.*

   Solution: A fixed student can sit with no more that $30-1=29$ other students, so the number of different seating arrangements satisfying the given conditions is at most 29. There are several ways to see there are indeed 29 different arrangements.

   Let the students be "named"$0,1,2,\ldots,28$ and ∞. Let the days be numbered 0 through 28. On day k, if k is even, student $k/2$ is seated with student ∞, and, for $i\in\{0,1,2,\ldots,28\}, i\neq k/2$, student i is seated with student $(k-i)$ mod 29. On day k, if k is odd, student $(k+29)/2$ is seated with student ∞, and, for $i\in\{0,1,2,\ldots,28\}, i\neq (k+29)/2$, student i is seated with student$(k-i)$ mod 29.

   In other words, for $i\neq j$, students i and j are seated together on day k if and only if $i+j\equiv k \pmod{29}$ and student i is seated with student ∞ on day $2i$ mod 29.

   Another seating can be determined geometrically. Place the numbers 0 through 28 evenly along the circumference of a circle and place ∞ at the center of the circle. On day k, student k will be seated with ∞ and students i and student j will be seated together if and only if, on the circle, the line joining i to j is perpendicular to the line joining k and ∞. There are 29 different lines joining ∞ to a number on the circumference of the circle. It follows that no student sits with the same student on two different days.

3. *Two balls of snow, one with diameter of 1 meter and the other with diameter of 10 meters are melting together under the action of the sun's rays. The first ball melts completely in one hour. How many hours will it take for the second ball to melt completely? Justify your answer. Note: For a ball of snow, the rate of melting at time t is proportional to the surface area of the ball.*

   Solution: Let V denote the volume of a ball of snow, A its surface area, and r its radius. We are given that $\dfrac{dV}{dt}=kA$, where k is some constant. Since $V=\dfrac{4}{3}\pi r^3$, and $A=4\pi r^2$, we have

   $$k\cdot A=\frac{dV}{dt}=\frac{dV}{dr}\cdot\frac{dr}{dt}=A\cdot\frac{dr}{dt}.$$

Therefore, $\frac{dr}{dt} = k$ and $r = kt + C$, where k and C are constants with k having the same value for all balls of snow.

For the small ball, $r = \frac{1}{2}$ when $t = 0$ and $r = 0$ when $t = 1$, Hence, the radius of the small ball at time t is given by $r = \frac{1}{2} - \frac{t}{2}$. Thus, the radius of the large ball at time t is given by $r = C_1 - \frac{t}{2}$, for some constant C_1. Since the radius of the large ball is 5 meters at time 0, $r = 5 - \frac{t}{2}$. It follows immediately that the large ball of snow melts after 10 hours.

4. *In $\triangle ABC$, let a denote the length of the side opposite the angle at A, b denote the length of the side opposite B, and c denote the length of the side opposite the angle at C. If the angle at A is $60°$, $a + b = p$, and $a + c = q$; find the value of a in terms of p and q.*

   Solution: Now $b = p - a$ and $c = q - a$. Therefore, by the Law of Cosines,

$$a^2 = (p - a)^2 + (q - a)^2 - 2(p - a)(q - a) \cdot \frac{1}{2}.$$

   This simplifies to $0 = p^2 + q^2 - pa - qa - pq$. Solving for a gives

$$a = \frac{p^2 - pq + q^2}{p + q}.$$

5. *Let N be a positive integer written in base 10. Let S be equal to the sum of the digits of N* **plus** *the sum of all products of the digits of N taken two at a time* **plus** *the sum of all the products of digits taken three at a time* **plus** ... **plus** *the product of all of the digits of N. Show that $S \leq N$. (For example, if $N = 123$, $S = 1 + 2 + 3 + 1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3 + 1 \cdot 2 \cdot 3$.) Describe the numbers, N, for which $S = N$.*

   Solution: If $N = a_0 + a_1 \cdot 10 + \cdots + a_n \cdot 10^n$, it is easy to check that S is equal to $(a_0 + 1)(a_1 + 1) \cdots (a_n + 1) - 1$. We must show that

$$(a_0 + 1)(a_1 + 1) \cdots (a_n + 1) - 1 \leq a_0 + a_1 \cdot 10 + \cdots + a_n \cdot 10^n$$

   provided $a_0, a_1, \cdots a_n$ are in $\{0, 1, \cdots, 9\}$.

   For $n = 0$ this result is clear. Let $n \geq 0$ and assume, by induction, that the inequality holds for n. Adding one to both sides of the inequality, then multiplying both sides by $a_{n+1} + 1$ and finally subtracting 1 from both sides gives,

$$(a_0 + 1)(a_1 + 1) \cdots (a_{n+1} + 1) - 1 \leq (a_0 + a_1 \cdot 10 + \cdots + a_n \cdot 10^n + 1)(a_{n+1} + 1) - 1.$$

   However,

$$\begin{aligned}(a_0 &+ a_1 \cdot 10 + \cdots + a_n \cdot 10^n + 1)(a_{n+1} + 1) - 1 \\ &= (a_0 + a_1 \cdot 10 + \cdots + a_n \cdot 10^n) + a_{n+1}(1 + a_0 + a_1 \cdot 10 + \cdots + a_n \cdot 10^n) \\ &\leq (a_0 + a_1 \cdot 10 + \cdots + a_n \cdot 10^n) + a_{n+1}(1 + 9 + 9 \cdot 10 + \cdots + 9 \cdot 10^n) \\ &= a_0 + a_1 \cdot 10 + \cdots + a_n \cdot 10^n + a_{n+1} \cdot 10^{n+1}\end{aligned}$$

   Hence the result is true for $n + 1$.

   Further, we see that equality holds only when

$$a_0 + a_1 \cdot 10 + \cdots + a_n \cdot 10^n = 9 + 9 \cdot 10 + \cdots + 9 \cdot 10^n,$$

   that is, when all of the digits of N, except possibly the left-most one, is 9.

6. *Let* $a_1, a_2, a_3, \ldots a_{2003}$ *be an arrangement of the integers* $1, 2, 3, \ldots 2003$.

   (a) *Determine the arrangement(s) for which*

   $$\frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_{2003}}{2003}$$

   attains its maximum value.

   (b) *Determine the arrangement(s) for which*

   $$\frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_{2003}}{2003}$$

   attains its minimum value.

   In both cases, prove your answer.

Solution: The following lemma can be verified in a straightforward manner.

Lemma: Suppose $1 \le i < j \le 2003$. Let $a_1, a_2, \ldots, a_{2003}$ be an arrangement of $1, 2, \ldots, 2003$. Let $b_1, b_2, \ldots b_{2003}$ be the arrangement in which a_i and a_j are interchanged. Let

$$S = \frac{a_1}{1} + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_{2003}}{2003}$$

and let

$$T = \frac{b_1}{1} + \frac{b_2}{2} + \frac{b_3}{3} + \cdots + \frac{b_{2003}}{2003}.$$

Then

$$S - T = \frac{a_i}{i} + \frac{a_j}{j} - \frac{a_j}{i} - \frac{a_i}{j} = \frac{(a_i - a_j)(j - i)}{ij}.$$

Continuation of Solution:

(a) Suppose the arrangement $a_1, a_2, \ldots, a_{2003}$ has been chosen so that the value S is maximal among all arrangements. Then, whenever $i < j$, the sum T, described above, is at most S. Hence $S - T \ge 0$. It follows from the lemma that $a_i > a_j$ whenever $i < j$. Therefore, the arrangement for which the maximum is attained is

$$2003, 2002, 2001, \ldots, 3, 2, 1.$$

The value of S for this arrangement is $2004 \sum_{i=1}^{2003} \frac{1}{i} - 2003$, which, according to Maple, is approximately 14,390.

(b) Suppose the arrangement $a_1, a_2, \ldots, a_{2003}$ has been chosen so that the value S is minimal among all arrangements. Arguing as above, we see that $a_i < a_j$ whenever $i < j$. Therefore, the arrangement for which the minimum is attained is

$$1, 2, 3, \ldots, 2001, 2002, 2003.$$

The value of S for this arrangement is clearly 2003.