Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
Illinois College March 28, 2003

Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution.

Enjoy the problem solving!

1. Suppose x and n are positive integers and that 2^n divides $x^{2003}+1$. Find, in terms of n, the smallest possible value of x.

2. At one time, school desks were made so that two students sat at each desk. For a class of 30 students and 15 desks, the teacher wanted to determine k different seating arrangements so that no pair of students sat together in more than one of the k seating arrangements. What is the largest possible value of k for which this can be done? Prove your answer.

3. Two balls of snow, one with diameter of 1 meter and the other with diameter of 10 meters are melting together under the action of the sun's rays. The first ball melts completely in one hour. How many hours will it take for the second ball to melt completely? Justify your answer. Note: For a ball of snow, the rate of melting at time t is proportional to the surface area of the ball.

4. In $\triangle ABC$, let a denote the length of the side opposite the angle at A, b denote the length of the side opposite B, and c denote the length of the side opposite the angle at C. If the angle at A is 60°, $a+b=p$, and $a+c=q$; find the value of a in terms of p and q.

5. Let N be a positive integer written in base 10. Let S be equal to the sum of the digits of N **plus** the sum of all products of the digits of N taken two at a time **plus** the sum of all the products of digits taken three at a time **plus** ... **plus** the product of all of the digits of N. Show that $S \le N$. (For example, if $N=123$, $S=1+2+3+1\cdot 2+1\cdot 3+2\cdot 3+1\cdot 2\cdot 3$.) Describe the numbers, N, for which $S=N$.

6. Let $a_1, a_2, a_3, \ldots a_{2003}$ be an arrangement of the integers $1, 2, 3, \ldots 2003$.

   (a) Determine the arrangement(s) for which

   $$\frac{a_1}{1}+\frac{a_2}{2}+\frac{a_3}{3}+\cdots+\frac{a_{2003}}{2003}$$

   attains its maximum value.

   (b) Determine the arrangement(s) for which

   $$\frac{a_1}{1}+\frac{a_2}{2}+\frac{a_3}{3}+\cdots+\frac{a_{2003}}{2003}$$

   attains its minimum value.

   In both cases, prove your answer.