

Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
McKendree College, April 5, 2002
Solutions

1. Solve the following equation for x :

$$((-1)^{\lfloor 2x \rfloor} x^2) + 4x + 1 = 0.$$

(Recall that $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .)

Solution:

Since $\lfloor 2x \rfloor$ is an integer, $(-1)^{\lfloor 2x \rfloor}$ is either plus one or minus one. There are thus two cases:

$\lfloor 2x \rfloor$ is even: In this case, $x^2 + 4x + 1 = 0$. Thus either $x = -2 + \sqrt{3}$ or $-2 - \sqrt{3}$. Consider first the case that $x = -2 + \sqrt{3}$. Since $9 < 12 < 16$, we have $3 < 2\sqrt{3} < 4$. Therefore, $\lfloor 2x \rfloor = \lfloor -4 + 2\sqrt{3} \rfloor = -1$ and, thus, $-2 + \sqrt{3}$ is not a root.

In a similar manner, if $x = -2 - \sqrt{3}$, $\lfloor 2x \rfloor = \lfloor -4 - 2\sqrt{3} \rfloor = -8$. Hence $-2 - \sqrt{3}$ is a root.

$\lfloor 2x \rfloor$ is odd: In this case, $x^2 - 4x - 1 = 0$. Therefore, $x = 2 \pm \sqrt{5}$. Since $16 < 20 < 25$, $4 < 2\sqrt{5} < 5$. Hence $\lfloor 2(2 + \sqrt{5}) \rfloor = \lfloor 4 + 2\sqrt{5} \rfloor = 8$ and $\lfloor 2(2 - \sqrt{5}) \rfloor = \lfloor 4 - 2\sqrt{5} \rfloor = 1$.

Thus, the roots to the given equation are $-2 - \sqrt{3}$ and $2 - \sqrt{5}$.

2. Can 2001, 2002, and 2003 be terms, not necessary consecutive, of a single geometric sequence? Either find an example of such a sequence or show that none exists.

Solution: No such sequence exists. Suppose, by way of contradiction, that such a sequence does exist. Thus, there exists, a, r , real numbers and, i, j, k , positive integers, such that $2001 = ar^i$, $2002 = ar^j$, and $2003 = ar^k$. If r is negative, then i, j, k must be even and thus, by replacing r by $-r$, we may assume r is positive. If r is between 0 and 1, by replacing r by $1/r$, i by $-i$, j by $-j$, and k by $-k$, we may assume that $2000 = ar^i$, $2001 = ar^j$, and $2002 = ar^k$, where $r > 1$ and i, j, k are integers. Since $r^i < r^j < r^k$ and $r > 1$, we have $i < j < k$. Letting $s = j - i$ and $t = k - j$, then $0 < s, 0 < t$ and

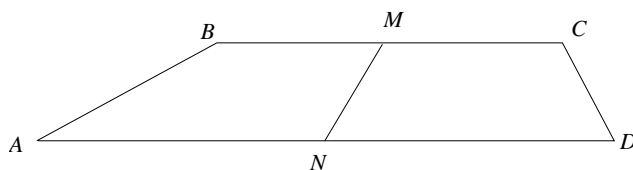
$$\frac{2002}{2001} = r^s \quad \frac{2003}{2002} = r^t.$$

Hence,

$$r^{st} = \left(\frac{2002}{2001} \right)^t = \left(\frac{2003}{2002} \right)^s,$$

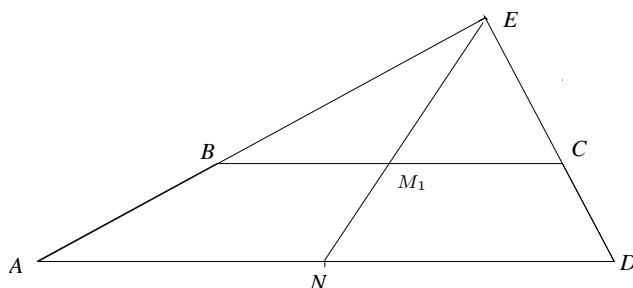
which in turn shows $2002^{s+t} = 2001^t 2003^s$. This is a contradiction since the left hand side of this equation is an even integer and the right hand side is an odd integer.

3. Suppose that $ABCD$ is a trapezoid with $BC \parallel AD$, M is the midpoint of BC , and N is the midpoint of AD , as shown.



Suppose $\angle BAD = 27.3^\circ$, $\angle ADC = 62.7^\circ$, $BC = 6$, and $AD = 10$. What is the length of segment MN ? Justify your answer.

Solution: Extend AB and CD until they meet at E . Join E to N and let M_1 be the point where this line meets BC .



Since $BC \parallel AD$, $\angle EM_1C = \angle END$ and $\angle ECM_1 = \angle EDN$. Therefore, since $\triangle EM_1C$ and $\triangle END$ share a common angle, they are similar. Hence $EM_1/EN = M_1C/ND$. Similarly, $\triangle EM_1B$ and $\triangle ENA$ are similar and $EM_1/EN = M_1B/NA$. Since $NA = ND$, it follows that $M_1B = M_1C$, i.e. M_1 and M are the same point. In particular, $EM/EN = MB/NA$.

Since $\angle DAE + \angle ADE = 90^\circ$, $\angle AED$ is a right angle. Therefore, A , E , and D are points on the circle whose diagonal is AD . Hence, $NA = ND = EN = 5$. Since $EM = 3$, the results of the last paragraph show that MB is also 3. Thus $MN = EN - EM = 2$.

4. Suppose a and b are relatively prime, positive integers with $a < b$. Let

$$\left\{ \frac{a_n}{b_n} \right\}_{n=1}^{\infty}$$

be the sequence of fractions, with positive numerators and denominators, such that $\frac{a_1}{b_1} = \frac{a}{b}$ and, for $n \geq 1$, $\frac{a_{n+1}}{b_{n+1}}$ is the **reduced** fraction equal to $\frac{a_n + 1}{b_n + 1}$.

- (a) Determine when $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists and find all possible values for it.
 (b) Determine when $\lim_{n \rightarrow \infty} (b_n - a_n)$ exists and find all possible values for it.

Solution: Both limits are always 1. This will follow from the following

Claim: Regardless of the values of a and b , $\lim_{n \rightarrow \infty} (b_n - a_n) = 1$.

Proof of Claim: Suppose a and b satisfy the hypotheses and that

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \dots, \frac{a_n}{b_n}, \dots$$

are defined as above. Note, for $n \geq 1$, that if $d = \gcd(a_n + 1, b_n + 1)$, then $a_{n+1} = (a_n + 1)/d$ and $b_{n+1} = (b_n + 1)/d$. Hence, $b_{n+1} - a_{n+1} = (b_n - a_n)/d \leq a_n - b_n$. Therefore, the difference between the numerator and denominator of $\frac{a_n}{b_n}$ is a non-increasing function of n and this difference decreases whenever the number d , defined above, is greater than 1. Thus, the sequence of differences is a non-increasing sequence of positive integers and therefore must converge. Hence, there are integers M and N such that for $n \geq N$, $b_n - a_n = M$, $a_{n+1} = a_n + 1$, and $b_{n+1} = b_n + 1$. Choose t , an integer, so that $tM - a_n$ is positive. Then

$$a_{n+tM-a_n} = a_n + tM - a_n = tM$$

and

$$b_{n+tM-a_n} = b_n + tM - a_n = (t+1)M.$$

Since every term of the original sequence is a reduced fraction, M must be one, as claimed.

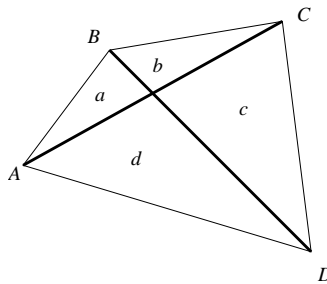
5. Thirteen is thought to be an “unlucky” number. A six-digit ticket number (which could start with one or more zeroes) will be called “lucky” if the sum of the squares of its first three digits equals the sum of the squares of its last three digits. Is the sum of all lucky ticket numbers divisible by 13 or not? Justify your answer.

Solution: Suppose the six-digit ticket number $ABCDEF$ is lucky. Then $DEFABC$ is also lucky. If ABC and DEF are different, we have found two different lucky ticket numbers. However, as numbers,

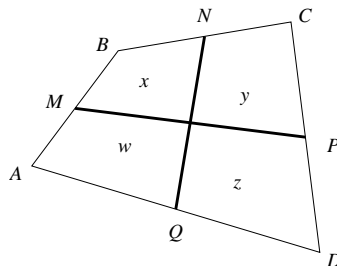
$$ABCDEF + DEFABC = (ABC + DEF)1000 + (ABC + DEF) = (ABC + DEF)(1001)$$

Since 13 divides 1001, 13 divides the sum of these two lucky ticket numbers. If ABC and DEF are the same, i.e. if the first three digits and the last three digits of the ticket number are the same, then, as a number, $ABCABC = (ABC)(1001)$. Hence 13 divides this lucky ticket number. Therefore 13 divides the sum of all lucky ticket numbers.

6. A “times” sign can be drawn on a quadrilateral $ABCD$ by drawing the diagonals AC and BD . This partitions the quadrilateral into four triangles whose areas will be denoted a , b , c , and d , as shown.



A “plus” sign can be drawn on a quadrilateral $ABCD$ by finding M , N , P , and Q , the midpoints of AB , BC , CD , and DA , respectively, and then drawing segments MP and NQ . This partitions the quadrilateral into four quadrilaterals whose areas will be denoted by x , y , z , and w , as shown.



- (a) Show that
- $a \times c = b \times d$
- .

Solution: Let F denote the intersection of the diagonals AC and BD .

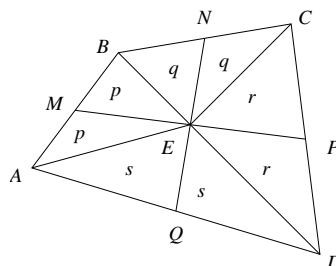
The areas of the four triangles can be expressed in terms of segments AF , BF , CF , and DF , and the angle α between the diagonals. Note that if $\angle AFB = \alpha$, then $\angle BFC = \pi - \alpha$, $\angle CFD = \alpha$, and $\angle DFA = \pi - \alpha$. Therefore,

$$\begin{aligned} a &= \frac{1}{2}AF \cdot BF \sin \alpha \\ b &= \frac{1}{2}BF \cdot CF \sin(\pi - \alpha) \\ &= \frac{1}{2}BF \cdot CF \sin \alpha \\ c &= \frac{1}{2}CF \cdot DF \sin \alpha \\ d &= \frac{1}{2}DF \cdot AF \sin(\pi - \alpha) \\ &= \frac{1}{2}DF \cdot AF \sin \alpha \end{aligned}$$

Thus,

$$ac = \frac{1}{4}AF \cdot BF \cdot CF \cdot DF \cdot \sin^2 \alpha = bd.$$

- (b) Show that
- $x + z = y + w$
- .

Solution: Let M , N , P and Q be the midpoints as defined, above. Let E be the point where the diagonals intersect and let lines joining E to the vertices be drawn, as shown below.

Now $\triangle BEM$ and $\triangle MEA$ share a common side. Since M is the midpoint of BA , the length of the perpendicular from B to MP is the same as the length of the perpendicular from A to MP , extended. Therefore, $\triangle BEM$ and $\triangle MEA$ have the same base and the same altitude, and thus, they have the same area, say p . Similarly, $\triangle BEN$ and $\triangle ENC$ have the same area, say q ; $\triangle CEP$ and $\triangle PED$ have the same area, say r ; and $\triangle DEQ$ and $\triangle EQA$ have the same area, say s . Since $x = p + q$, $y = q + r$, $z = r + s$, and $w = s + p$; we see that $x + z = y + w = p + q + r + s$.