Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
McKendree College, April 5, 2002

Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution.

Enjoy the problem solving!

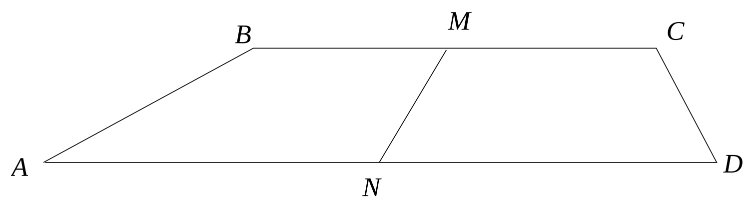
1. Solve the following equation for x:

$$((-1)^{\lfloor 2x \rfloor} x^2) + 4x + 1 = 0.$$

(Recall that $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

2. Can 2001, 2002, and 2003 be terms, not necessary consecutive, of a single geometric sequence? Either find an example of such a sequence or show that none exists.

3. Suppose that $ABCD$ is a trapezoid with $BC \| AD$, M is the midpoint of BC, and N is the midpoint of AD, as shown.



Suppose $\angle BAD = 27.3°$, $\angle ADC = 62.7°$, $BC = 6$, and $AD = 10$. What is the length of segment MN? Justify your answer.

4. Suppose a and b are relatively prime, positive integers with $a < b$. Let

$$\left\{ \frac{a_n}{b_n} \right\}_{n=1}^{\infty}$$

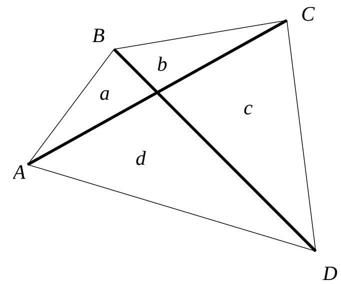
be the sequence of fractions, with positive numerators and denominators, such that $\frac{a_1}{b_1} = \frac{a}{b}$ and, for $n \geq 1$, $\frac{a_{n+1}}{b_{n+1}}$ is the **<u>reduced</u>** fraction equal to $\frac{a_n + 1}{b_n + 1}$.

(a) Show that $\lim_{n \to \infty} \frac{a_n}{b_n}$ exists and find all possible values for it.

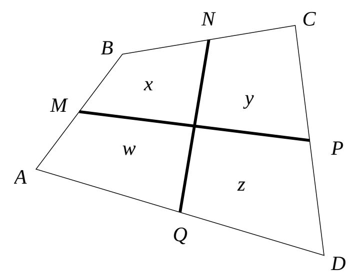
(b) Show that $\lim_{n \to \infty} (b_n - a_n)$ exists and find all possible values for it.

5. Thirteen is thought to be an "unlucky" number. A six-digit ticket number (which could start with one or more zeroes) will be called "lucky" if the sum of the squares of its first three digits equals the sum of the squares of its last three digits. Is the sum of all lucky ticket numbers divisible by 13 or not? Justify your answer.

6. A "times" sign can be drawn on a quadrilateral $ABCD$ by drawing the diagonals AC and BD. This partitions the quadrilateral into four triangles whose areas will be denoted a, b, c, and d, as shown.



A "plus" sign can be drawn on a quadrilateral $ABCD$ by finding M, N, P, and Q, the midpoints of AB, BC, CD, and DA, respectively, and then drawing segments MP and NQ. This partitions the quadrilateral into four quadrilaterals whose areas will be denoted by x, y, z, and w, as shown.



(a) Show that $a \times c = b \times d$.

(b) Show that $x + z = y + w$.