# Student Mathematics Competition of the Illinois Section of the MAA
## University of Illinois – March 23, 2001 – Solutions

1. A total of 2001 lines, called "boundaries", are drawn in the plane. These boundaries divide the plane into regions, called "countries". Some of the countries are infinite.

   (a) A train track in the shape of a straight line segment is constructed so that it does not pass through any point where two boundaries intersect. What is the maximum number of countries the train track can pass through?

   (b) What is the maximum number of countries a circular train track can pass through, provided the track does not pass through any point where two boundaries intersect?

   **Solution (a):** The track meets each line at most one time and thus there are at most 2001 points where the track meets a boundary. These points divide the track into at most 2002 intervals. The interior of each interval belongs to a single country. Thus the track meets at most 2002 countries. (This maximum is attainable in every system of boundary lines.)

   **Solution (b):** If the track is circular, it can meet each boundary at most twice and thus there are at most 4002 points where the track meets a boundary. These points determine at most 4002 arcs each belonging to at most one country. (This maximum is also attainable in every system of boundary lines.)

2. Let $a$, $b$, $c$ be the sides of a triangle. Show

$$(a+b-c)(a-b+c)(-a+b+c) \leq abc.$$

   **Solution:** Let $A = a+b-c$, $B = a-b+c$, and $C = -a+b+c$. The numbers $A$, $B$, and $C$ are positive. It is well-known that if $x$ and $y$ are positive then $\sqrt{xy} \leq \frac{x+y}{2}$. Thus,

$$(a+b-c)(a-b+c)(-a+b+c) = ABC = \sqrt{AB}\sqrt{BC}\sqrt{CA} \leq \frac{A+B}{2} \cdot \frac{B+C}{2} \cdot \frac{C+A}{2} = acb.$$

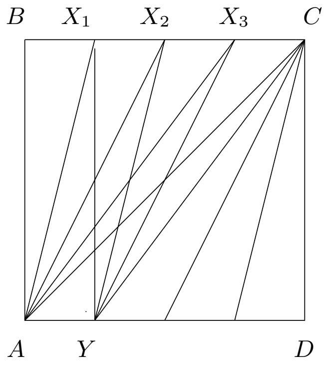
3. Suppose $ABCD$ is a square and $n$ is a positive integer. Let $X_1, X_2, \ldots, X_n$ be points on $BC$ so that

$$BX_1 = X_1X_2 = \cdots = X_{n-1}X_n = X_nC.$$

   Let $Y$ be a point of $AD$ so that $AY = BX_1$. Find (in degrees) the value of

$$\angle AX_1Y + \angle AX_2Y + \cdots + \angle AX_nY + \angle ACY.$$

   **Solution:** The value of this sum is 45°. Proof without words.



4. Suppose $A$ is a positive integer and $B = A^3$. It is possible that number of digits in $A$ plus the number of digits in $B$ equals 2001.

   **Solution:** Suppose $A$ is a $k$ digit number. Hence, $10^{k-1} \le A < 10^k$. Thus, $10^{3k-3} \le B < 10^{3k}$ and $B$ is either a $3k$, $3k-1$, or $3k-2$ digit number. The number of digits of $A$ plus the number of digits of $B$ is of the form $4k$, $4k-1$, or $4k-2$ and 2001 is not of any of these forms.

5. For $x$ a positive integer, define the sequence

$$x_1, x_2, x_3, \ldots$$

   by $x_1 = x$ and, for $j \ge 2$, $x_j$ is twice the sum of the digits of $x_{j-1}$.

   (a) Show that if $n > 1$ is an integer and $x = 2^n$, then the sequence $x_1, x_2, x_3, \ldots$ contains a one digit number.

   (b) Show that if $n > 2$ is an integer and $x = 3^n$, then the sequence $x_1, x_2, x_3, \ldots$ does not contains a one digit number.

   **Solution:** For $y$ a positive number, let $S(y)$ denote twice the sum of the digits of $y$. Observe first that $S(y) \equiv 2y \pmod 9$ since $y$ and the sum of its digits are congruent modulo 9.

   Consider the sequence $y_1, y_2, y_3, \ldots$ defined by $y$. It is easy to show that if $y_i$ has more than 2 digits, then $y_{i+1} < y_i$. If $y_i$ is a two digit number, then $y_{i+1} \le 18$. Since $y_2, y_3, \ldots$ are all even, we have, for some $i$, $y_i$ has a value given in the following table (which also shows the value of $y_{i+1}$ in each case).

| $y_i$ | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|---|
| $y_{i+1}$ | 4 | 8 | 12 | 16 | 2 | 6 | 10 | 14 | 18 |

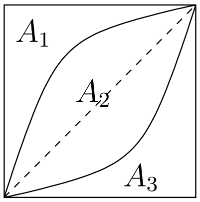
   Thus every sequence either contains a one digit number (which is not congruent to zero modulo 9) or eventually stabilizes at 18. Note that these two possibilities cannot both occur for the same sequence.

   If $y$ is divisible by 9, the sequence eventually will go to the fixed point 18. If $y$ is not divisible by 9, the sequence will contain a one digit number. Parts (a) and (b) follow immediately.

6. Evaluate $\int_0^2 (\sqrt{1+x^3} + \sqrt[3]{x^2+2x})\ dx$.

   **Solution:** Let $f$ be the function defined by $f(x) = \sqrt{1+x^3} - 1$. It is easy to show $f(0) = 0$; $f(2) = 2$; $f(x) \le x$, $x \in [0,2]$; and $f^{-1}(x) = \sqrt[3]{x^2+2x}$. In particular, the graph of $y = f(x)$ lies below the line $y = x$ for $x \in [0,2]$ and the graph of $y = f^{-1}(x)$ (obtained from reflecting the graph of $y = f(x)$ in the line $y = x$) lies above $y = x$ for $x \in [0,2]$.

   Let $A_1$, $A_2$, and $A_3$ are the areas of the three regions determined by the curves $y = f(x)$, $y = f^{-1}(x)$, $y = 0$, $y = 2$, $x = 0$, $x = 2$, as shown:



   Since $A_1 = A_3$,

$$4 = A_1 + A_2 + A_3 = A_3 + (A_2 + A_3) = \int_0^2 f(x)\ dx + \int_0^2 f^{-1}(x)\ dx.$$

   It now follows easily that $\int_0^2 (\sqrt{1+x^3} + \sqrt[3]{x^2+2x})\ dx = 6$.