Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
University of Illinois, March 23, 2001

Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution. Enjoy the problem solving.

1. A total of 2001 lines, called "boundaries", are drawn in the plane. These boundaries divide the plane into regions, called "countries". Some of the countries are infinite.
   (a) A train track in the shape of a straight line segment is constructed so that it does not pass through any point where two boundaries intersect. What is the maximum number of countries the train track can pass through?
   (b) What is the maximum number of countries a circular train track can pass through, provided the track does not pass through any point where two boundaries intersect?

2. Let a, b, c be the sides of a triangle. Show

$$(a+b-c)(a-b+c)(-a+b+c) \le abc.$$

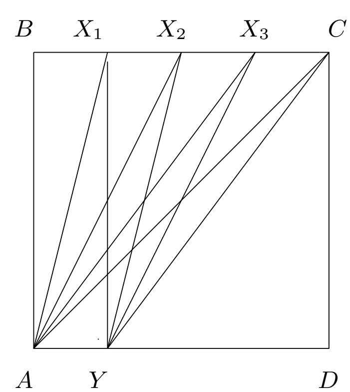
3. Suppose $ABCD$ is a square and n is a positive integer. Let $X_1, X_2, \ldots, X_n$ be points on BC so that

$$BX_1 = X_1X_2 = \cdots = X_{n-1}X_n = X_nC.$$

   Let Y be a point of AD so that $AY = BX_1$. Find (in degrees) the value of

$$\angle AX_1Y + \angle AX_2Y + \cdots + \angle AX_nY + \angle ACY.$$

   For example, if $n = 3$, the points are situated as shown in the following diagram.



4. Suppose A is a positive integer and $B = A^3$. It is possible that the number of digits in A plus the number of digits in B equals 2001?

5. For x a positive integer, define the sequence

$$x_1, x_2, x_3, \ldots$$

by $x_1 = x$ and, for $j \geq 2$, x_j is twice the sum of the digits of x_{j-1}.

(a) Show that if $n > 1$ is an integer and $x = 2^n$, then the sequence $x_1, x_2, x_3, \ldots$ contains a one digit number.

(b) Show that if $n > 2$ is an integer and $x = 3^n$, then the sequence $x_1, x_2, x_3, \ldots$ does not contains a one digit number.

6. Evaluate

$$\int_0^2 (\sqrt{1+x^3} + \sqrt[3]{x^2+2x})\ dx.$$