Solutions
Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
North Central College, March 31, 2000

Do any four of the six problems. Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution. Enjoy the problem solving.

1. *You roll 2000 dice one time and find that the ratio of the sum of the numbers on the top faces to the sum of the numbers of the bottom was again an integer. How many different integers are possible for the sum of the numbers on the top faces? Justify your answer*

   Solution: The number on the opposite side of a die always add to seven. Thus, if n is the sum of the numbers on the bottoms of the dice, $14000 - n$ is the number on the top. Since n is between 1 and 6, we must find all integers n such that $\dfrac{14000}{n}$ is an integer and n is between 2000 and 12000. Clearly, the number of different values of n is the same as the number different values of $k = \frac{14000}{n}$, But k can be any integer which divides 14000 and which is between $\frac{14000}{12000}$ and $\frac{14000}{1000}$, inclusive. There are six such integers — 2,4,5,and 7.

2. *Prove that if $x_1, x_2, \ldots, x_n$ are positive, real numbers, then*

$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{n-1} + \frac{1}{x_n}\right)\left(x_n + \frac{1}{x_1}\right) \geq 2^n.$$

   Solution: Note that if a and b are positive, real numbers, then $a + b \geq 2\sqrt{ab}$. Applying this to the given expression yields,

$$\begin{aligned}\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{n-1} + \frac{1}{x_n}\right)\left(x_n + \frac{1}{x_1}\right) &\geq 2^n\sqrt{\frac{x_1}{x_2}\frac{x_2}{x_3}\cdots\frac{x_n}{x_1}} \\ &= 2^n.\end{aligned}$$

3. *A snail is crawling in the garden. It starts at a certain point and crawls 15 minutes in one direction. Then it makes a 90° turn to either the left or right and crawls 15 minutes in this new direction. The snail repeats this process of crawling and turning until it returns to the starting point. Show that snail is at the starting point after a whole number of hours.*

   Solution: Orient the snail so that it goes either east, west, north or south. By a "crawl" will be meant a crawl of 15 minutes. The number of crawls equals the number of turns.

   Since the snail returns to the starting point, the number of east crawls equals the number of west crawls and the number of north crawls equals the number of south crawls. Finally, the number of east/west crawls equals the number of north/south crawls since each east/west crawl is followed by a north/south crawl. Hence the number of turns is equal to four times the number of east crawls.

4. *Evaluate* $\lim_{x\to 0} \frac{x^2 \int_0^x \sqrt{t^3+1}\ dt}{\int_0^x t^2 \cos t\ dt}$.

Solution: Since $|x|^2 = x^2$, the given limit is $\lim_{x\to 0} \frac{x^2 \int_0^x \sqrt{t^3+1}\ dt}{\int_0^x t^2 \cos t\ dt}$. This limit is of the form $\frac{0}{0}$ and thus, if the limit exists, L'Hopital's rule can be applied. Hence, if the limit exists, the Fundamental Theorem of Calculus show that the limit is equal to

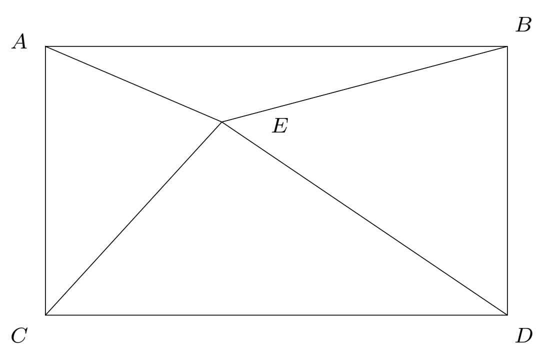
$$\lim_{x\to 0} \frac{x^2\sqrt{x^3+1} + 2x \int_0^x \sqrt{t^3+1}\ dt}{x^2 \cos x}.$$

If the limits of each summand exist, this limit is equal to

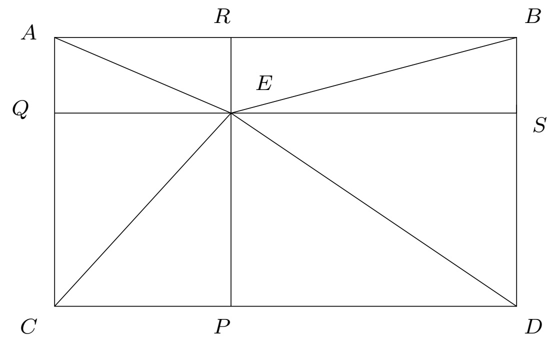
$$\begin{aligned} \lim_{x\to 0} \frac{x^2\sqrt{x^3+1}}{x^2\cos x} + \lim_{x\to 0} \frac{2x\int_0^x \sqrt{t^3+1}\ dt}{x^2 \cos x} &= \lim_{x\to 0} \frac{\sqrt{x^3+1}}{\cos x} + \lim_{x\to 0} \frac{2\int_0^x \sqrt{t^3+1}\ dt}{x\cos x} \\ &= 1 + \lim_{x\to 0} \frac{2\int_0^x \sqrt{t^3+1}\ dt}{x\cos x}. \end{aligned}$$

The limit in the last expression is of the form $\frac{0}{0}$ and thus if the limit exists, L'Hopital's Rule shows that it equals $\lim_{x\to 0} \frac{2\sqrt{x^3+1}}{\cos x - x\sin x}$. Since this limit is 2, the original limit is equal to 3.

5. *Suppose $ABCD$ is a rectangle and E is a point on the interior of the rectangle so that $AE = 1$, $CE = 2$ and $DE = 3$. Find all possible lengths for BE and justify your answer(s).*



Solution: Drop perpendiculars from E to each of the sides and let P, Q, R, S, be the points where these perpendiculars meet the sides, as shown below.



By the Pythagorean Theorem,

$$BE^2 + CE^2 = RB^2 + RE^2 + PC^2 + PE^2.$$

Since $RB = PD$ and $PC = RA$,

$$BE^2 + CE^2 = PD^2 + RE^2 + RA^2 + PE^2 = DE^2 + AE^2.$$

Hence, $BE^2 = 9 + 1 - 4$ and $BE = \sqrt{6}$.

6. *Let n be a positive number. Show there are a finite number of solutions to the equation*

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n} = 1,$$

where $x_1, x_2, x_3, \ldots, x_n$ are positive integers.

Solution: We show by induction that on n that if r is a fixed positive real, then there are only a finite number of solutions to

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n} = r,$$

with $x_1, x_2, x_3, \ldots, x_n$ a non-decreasing sequence of positive integers.

If n equals 1, the result is clear. Suppose for some value of k that for r fixed real number, the equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_k} = r,$$

where $x_1, x_2, x_3, \ldots, x_k$ are positive integers and

$$x_1 \leq x_2 \leq \cdots \leq x_k,$$

has only a finite number of solutions. Consider now an equation of the form

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_{k+1}} = s,$$

where s is a real number and x_1, x_2, $\ldots x_{k+1}$ are to be a non-decreasing sequence of positive integers. If we have a solution to such an equation, then

$$s = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_{k+1}} \leq \frac{k+1}{x_1},$$

Hence, $x_1 \leq \frac{k+1}{s}$. In particular, there are a finite number of choices for x_1. The result follows since, by mathematical induction, there are a finite number of solutions

$$\frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_{k+1}} = s - \frac{1}{x_1},$$

with $x_2, x_3, \ldots, x_{k+1}$ a non-decreasing sequence of positive integers.