Persistently Foliar Knots

Charles Delman (EIU)

Joint work with Rachel Roberts (WSTLU)

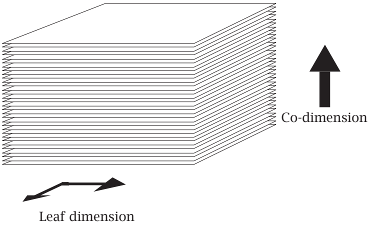
May 25, 2019

Georgia Topology Conference

Contents

Foliations

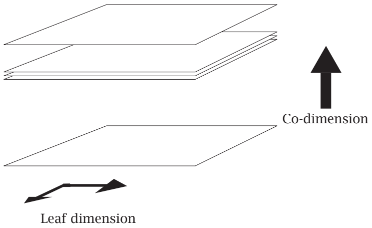
A *foliation* is a decomposition of a manifold into *leaves* of lower dimension. Locally, we have charts $\mathbb{R}^m \times \mathbb{R}^n$, with transitions that preserve the horizontal levels $\mathbb{R}^m \times \{y\}$.



We consider foliations of smooth 3-manifolds with 2-dimensional C^1-embedded leaves (co-dimension 1).

Laminations

If we replace $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^m \times C$, where $C \subseteq R$ is closed, the resulting collection of immersed leaves is called a *lamination*.



Key example: minimal set of a foliation.

Taut Foliations

Charles Delman (EIU)
Joint work with Rachel Roberts (WSTLU)

Definition

A co-dimension 1 foliation of a compact 3-manifold is *taut* if there is a 1-manifold transversely intersecting every leaf. We exclude the one example with a sphere leaf, $S^1 \times S^2$.

A closed, orientable manifold admitting a taut foliation is universally covered by $\mathbb{R}^3$, hence is irreducible and has infinite fundamental group.

Definition

A 3-manifold is *foliar* if it admits a co-orientable, taut (co-dimension 1) foliation (CTF).

Foliations in Knot Complements

Definition

A foliation $\mathcal{F}$ *strongly realizes* a slope in $\partial N(k)$ if $\mathcal{F}$ intersects $\partial N(k)$ transversely in a foliation by curves of that slope.

If a rational slope r is strongly realized by a co-oriented taut foliation $\mathcal{F}$, then capping off the leaves of $\mathcal{F}$ by disks produces a taut co-oriented foliation of the manifold obtained by surgery with coefficient r.

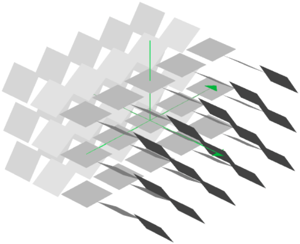
Definition

A knot k is *persistently foliar* if every real (i.e. $\neq \frac{1}{0}$) slope in $\partial N(k)$ is strongly realized by a co-oriented taut foliation.

If a knot k is persistently foliar, then every manifold obtained by non-trivial Dehn surgery on k is foliar.

Contact Structures

- A *contact structure* on a 3-manifold is a nowhere-integrable plane field.
- Locally, every contact structure looks like this:



Foliations Yield Pairs of Tight Contact Structures

Charles Delman (EIU)
Joint work with Rachel Roberts (WSTLU)

- But globally, contact structures fall into two categories: A contact structure is *tight* if there is no embedded disk tangent to the contact structure along its boundary circle; otherwise, it is *overtwisted*.
- The tangent planes of a taut foliation can be perturbed to two tight contact structures, one that twists positively and one that twists negatively, by results of Eliashberg-Thurston, Kazez-Roberts, and Bowden.
- The presence of these tight contact structures has homological ramifications.

Heegaard-Floer Homology

- Powerful invariant, introduced by P. Ozsváth & Z. Szabó.
- For rational homology 3-spheres, $\widehat{HF}(M)$ is a vector space over $\mathbb{F}_2$.
- $\text{Rank}(\widehat{HF}(M)) \geq |H_1(M, \mathbb{Z})|$.
- If equality holds, M is called an *L-space*.
- Ozsváth- Szabó showed that elliptic manifolds, such as lens spaces, are L-spaces (hence the name).
- Ozsváth- Szabó also showed the contact structures arising from a foliation introduce non-trivial elements in $\widehat{HF}(M)$, showing that M is not an L-space. Thus we have:

Theorem (E-Th, K-R, B, O-Sz)

If M is foliar, then M is not an L-space.

Questions & Conjectures

- Does the converse hold for irreducible 3-manifolds? (Ozsváth–Szabó, Boyer-Gordon-Watson, Juhasz)
- Restricting attention to surgery on knots $k \subset S^3$, call k an *L-space knot* if some non-trivial Dehn surgery on k yields an L-space. We conjecture the following:

  L-space Knot Conjecture *A knot k is persistently foliar if and only if it is not an L-space knot and has no reducible surgeries.* (Note that $\Rightarrow$ follows from results above. Only $\Leftarrow$ is conjectural.)

  More generally,

  L-space Surgery Conjecture *A manifold obtained by Dehn surgery on k is foliar if and only if it is irreducible and not an L-space.* (This could be strengthened to ask if such surgery slopes are strongly realized by a CTF.)

Summary So Far

- Foliar manifolds are not "simple." In particular, they are not L-spaces – rational homology spheres with minimal HF homology. Taut co-orientable foliations, via contact geometry, detect HF homology.
- Does HF homology detect foliations?
- In particular, does it detect foliations for manifolds obtained by surgery on knots in the 3-sphere, and are foliar slopes strongly realized, meaning that foliations are obtained by capping off foliations in the knot complement?
- Recall that a knot k is *persistently foliar* if all slopes are strongly realized by a CTF; hence k has no reducible or L-space surgeries.
- Is the converse true?
- Questions?

Results for Prime Knots

Theorem (D-Roberts)

If k is a prime alternating knot other than a $(2, n)$-torus knot, then k is persistently foliar.

Theorem (D-Roberts)

If k is a Montesinos knot with no L-space surgeries, then k is persistently foliar.

Remark

All L-space knots are fibered [Ghiggini & Ni]. In the case of alternating and Montesinos knots, the only L-space knots are torus knots and $(-2, 3, n)$-pretzel knots (n positive) [Baker, Lidman, Moore], which are easily seen directly to be fibered.

Results for Composite Knots

Composite knots are never L-space knots [Krcatovich]. We have proven the following:

Theorem (D-Roberts)

Any composite knot with a persistently foliar summand is persistently foliar.

Theorem (D-Roberts)

The connected sum of two fibered knots is persistently foliar.

Corollary

If each of at least two summands of a composite knot is either fibered, alternating, or Montesinos, then it is persistently foliar.

Knots Satisfying the L-space Knot Conjecture

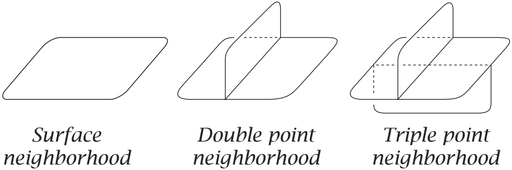
Charles Delman (EIU)
Joint work with Rachel Roberts (WSTLU)

Thus, the following knots satisfy the L-space Knot Conjecture:

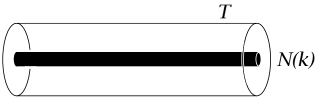
- Alternating knots.
- Montesinos knots.
- Composite knots with a persistently foliar summand.
- Composite knots in which each of two summands is fibered, alternating, or Montesinos.

Finite Depth Spines

- Build a *spine* [Casler] from a finite succession of transversely intersecting surfaces.
- Locally:

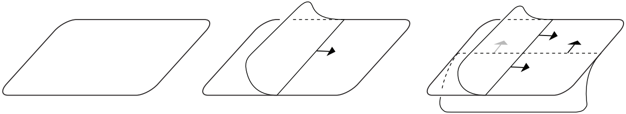


- Include a tube T parallel to $\partial N(k)$ in the spine.



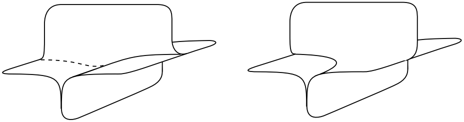
Smoothing Instructions $\to$ Transversely Oriented Branched Surface

- Successively introduce *smoothing instructions* at singular points, via compatible transverse orientations, to obtain a transversely orientable *branched surface* :



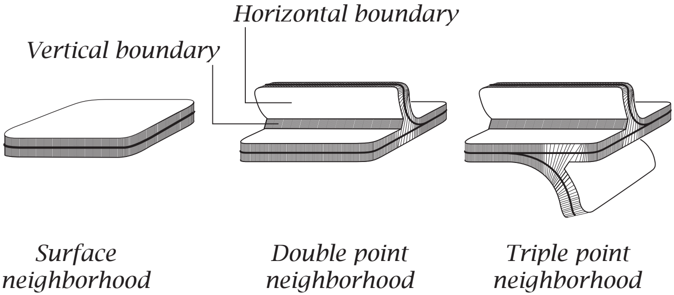
Surface neighborhood | *Double point neighborhood* | *Triple point neighborhood*

- It is often useful to think of a triple point this way:



Laminar Branched Surface Extends to Foliation

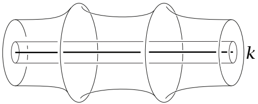
- Eventually obtain a transversely orientable *laminar* branched surface [Li] for which the complement of an I-bundle neighborhood is a taut sutured manifold.



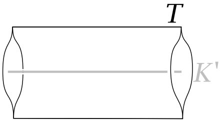
- "Infinite end" condition allows extension to foliation.

Persistence

- Want the complementary region containing $\partial N(k)$ to be $T^2 \times I$ with a positive even number of meridional cusps:

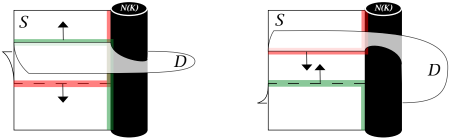


- This region may be filled with leaves meeting the boundary in a foliation by curves of any slope.
- After rational filling, this region is a solid torus with a positive and even number of longitudinal cusps, and may be filled with "stacked chairs."



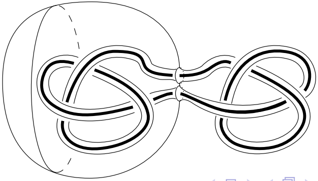
Historical Antecedents

- Sutured manifold theory for constructing finite depth foliations. [Gabai]



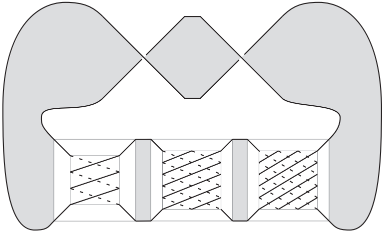
Sutured manifold decomposition of a Seifert surface

- "Swallow-follow" closed (branched) surface. [Menasco; Oertel]



Spines from Tangles & Murasugi Sums
Example: Montesinos Knots

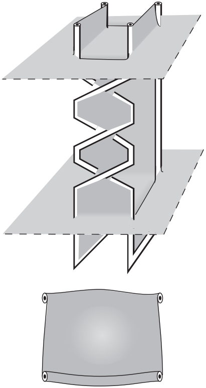
- A *Montesinos knot* is a knot whose diagram consists of rational tangles joined in a ring.

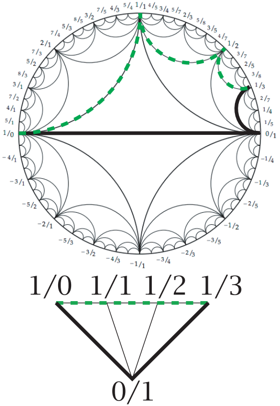


K(1/3, 2/5, 3/5, -2)

- Essential surfaces in rational tangles consist of plumbed bands and correspond to minimal paths in the Farey complex. [Hatcher - Thurston]
- Separate by spheres at plumbing levels.

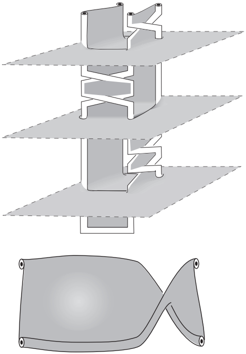
Surface & Path 1

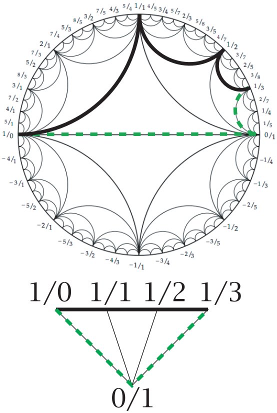




- From outside the tangle, we see an untwisted band.

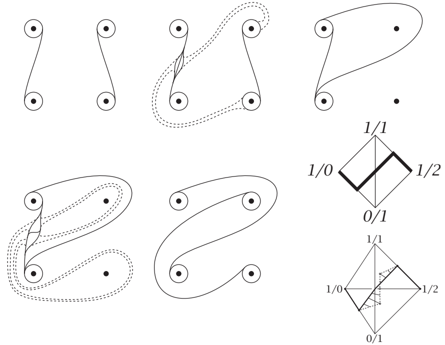
Surface & Path 2





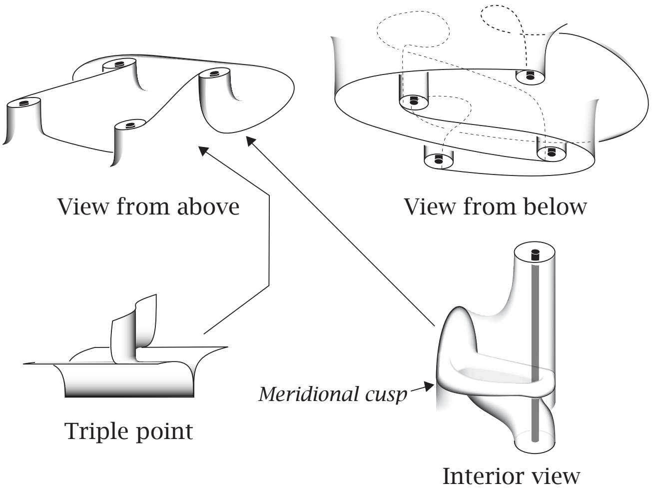
- From outside the tangle, we see a twisted band.

Combine Them! Here's the Transition & Path.

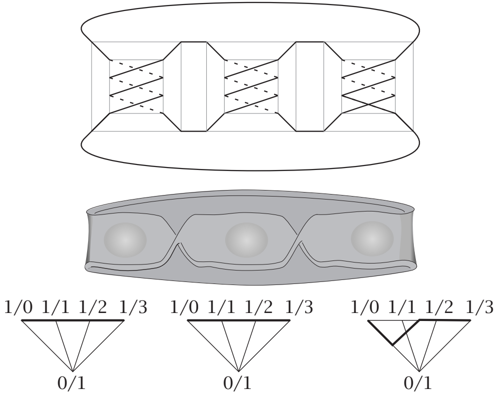


- Obtain two meridional cusps!
- Call the transition path a *channel*.
- Has potential to be generalized to Murasugi sums besides plumbing and tangles besides rational ones.

Channel Branched Surface (Locally)



The Enveloping Surface Complement

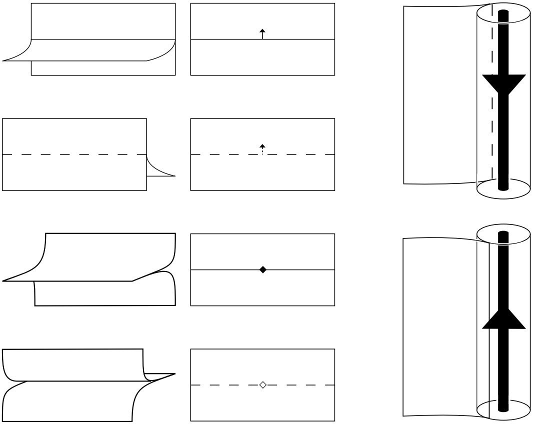


- The $(3, 3, 3)$-pretzel knot is persistently foliar!
- Analysis of Farey complex handles most Montesinos knots.

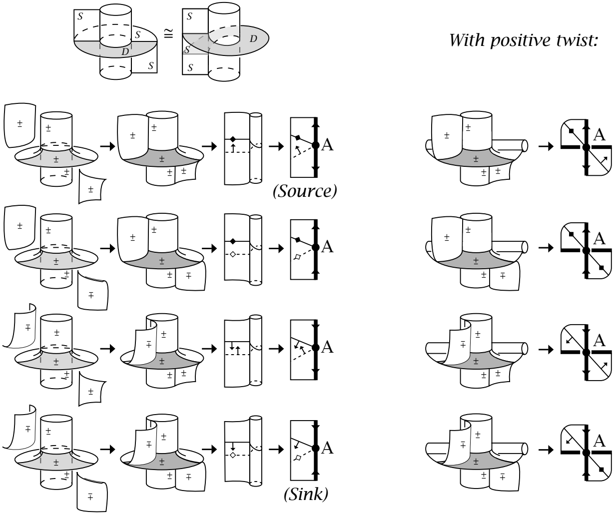
Spine Decompositions with Local Smoothing

- But not all.
- For the "recalcitrant" pretzel knots, alternating knots, and composite fibered knots, we introduce a step-by-step decomposition process.
- Much like sutured manifold decomposition, but by working at the spine level and smoothing locally as we proceed, we have many more choices at triple points.
- Arcs of decomposing surface lie on boundary of existing "surface complement" or on tube T; call arcs on T *transition arcs*.

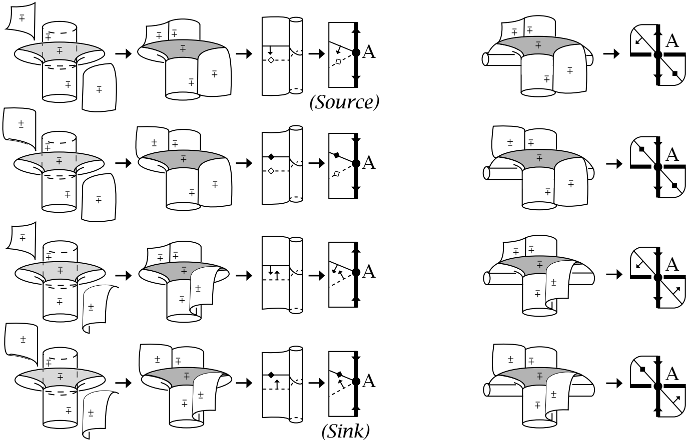
Notation Conventions



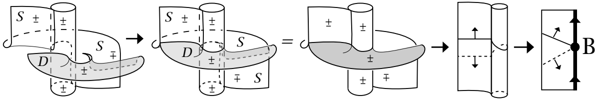
Local Models: Type A



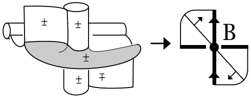
Local Models: Type A

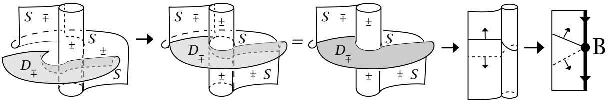


Local Models: Type B

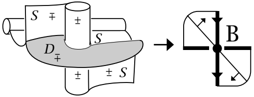


With positive twist:



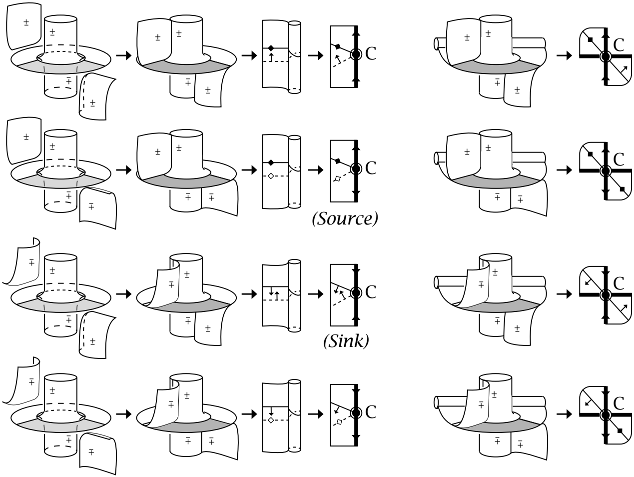


With positive twist:



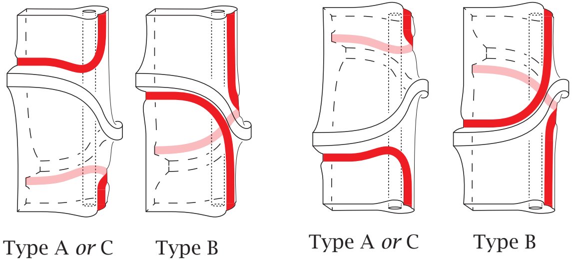
Local Models: Type C
Introducing Meridional Cusps

With positive twist:

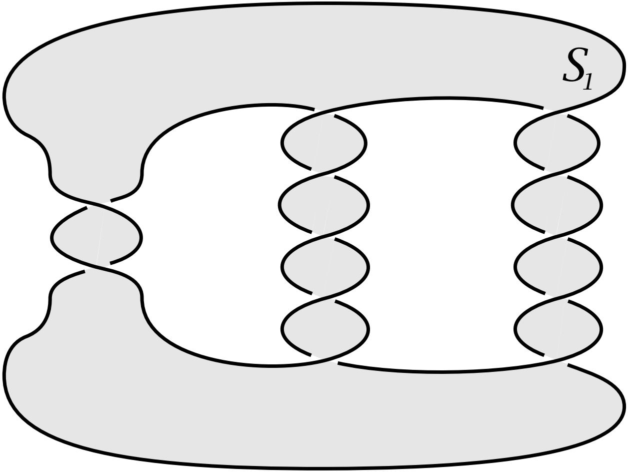


Behavior of Sutures at Transition Arcs

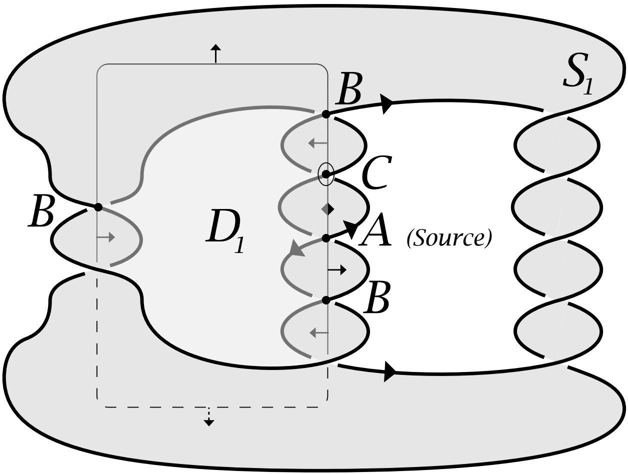
At transition arcs, sutures behave as in sutured manifold decompositions:



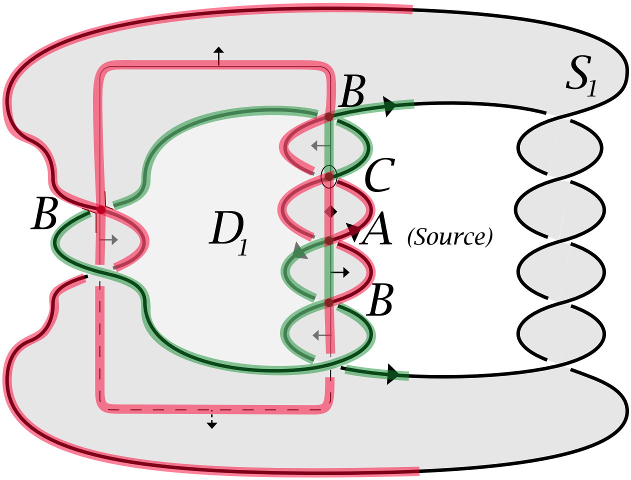
The $(-2, 5, 5)$ Pretzel Knot is Persistently Foliar



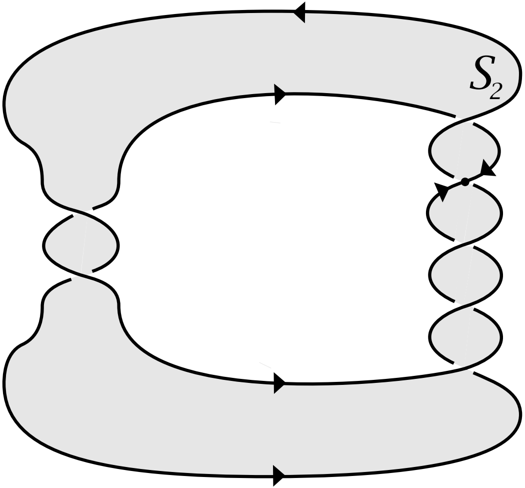
The $(-2, 5, 5)$ Pretzel Knot is Persistently Foliar



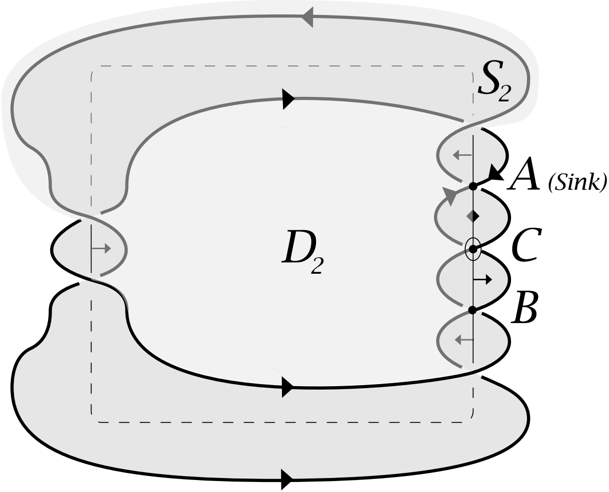
The $(-2, 5, 5)$ Pretzel Knot is Persistently Foliar



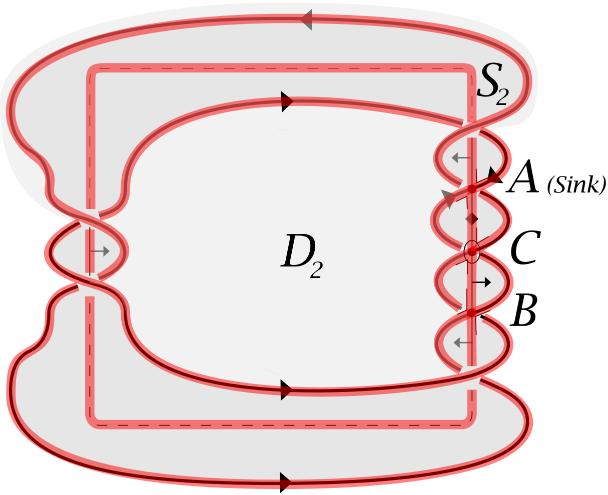
The $(-2, 5, 5)$ Pretzel Knot is Persistently Foliar



The $(-2, 5, 5)$ Pretzel Knot is Persistently Foliar

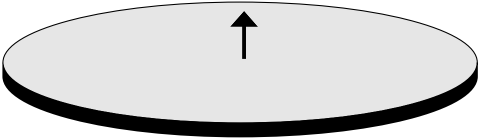


The $(-2, 5, 5)$ Pretzel Knot is Persistently Foliar



The $(-2, 5, 5)$ Pretzel Knot is Persistently Foliar

Charles Delman (EIU) *Joint work with* Rachel Roberts (WSTLU)



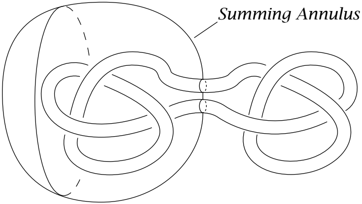
Alternating and Fibered Knots

- In general, for pretzel knots, we begin with the planar band surface.
- For alternating knots, we begin with the surface given by Seifert's algorithm on a reduced alternating projection.
- For fibered knots we begin with the fibering surface and decompose with a product disk.
- Prime fibered knots with monodromy that is neither right- nor left-veering are already known to be persistently foliar, but our approach yields new foliations.
- For composite fibered knots, regardless of monodromy, we obtain one meridional cusp in each summand.

A composite knot with a persistently foliar summand is persistently foliar.

Proposition

Any slope along a summand of a composite knot that is strongly realized is strongly realized along the sum.



Proof: On the summing annulus, align the foliation realizing the slope in the summand with a longitudinal foliation (which always exists by Gabai's proof of Property R) of the other summand.

Composite Knots

Charles Delman (EIU)
Joint work with Rachel Roberts (WSTLU)

- The advantage of composite knots is that, even if no summand is persistently foliar, we need only one meridional cusp around each of two summands for the knot to be persistently foliar.
- This approach should be widely generalizable, so we propose:

Conjecture (D-Roberts)

All composite knots are persistently foliar.

Work in Progress; Question for the Future

Work in progress:

- More general conditions that allow for the construction of co-oriented taut foliations that strongly realize all boundary slopes except one.
- More general Murasugi sums.
- More general tangles.

Question for the future:

- It would be really exciting to understand topological/geometric/structural links leading from Heegaard-Floer homology to CTFs!

Questions?

Thank You!