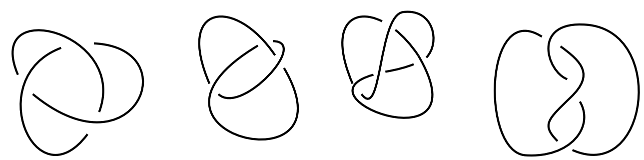
Elementary Calculation Methods for the Cord Ring of a Knot

October 16, 2009

Knots: a Reminder
Classical Knots

- A classical knot is simply an embedding of a circle, S^1, in compactified 3-dimensional space, S^3.



Equivalent Knots

- Two knots are equivalent if one can be continuously deformed into the other via embeddings (ambient isotopy).

Generalizations

- More generally, a knot is an embedding of any sphere S^n in a sphere whose dimension is two greater, S^{n+2}.
- Even more generally, one may consider manifold embeddings $M^n \to M^{n+2}$.
- Co-dimension two allows for interesting and very complicated global behavior.
- We will restrict our attention to classical knots, about which much still remains undiscovered. However, many of the ideas have natural generalizations.

Invariants

- A knot invariant is any object (such as a group, ring, polynomial, integer, or topological space) associated to the knot that is well-defined with respect to knot equivalence.

- One hopes to find invariants that:
  - Are reasonable to compute.
  - Distinguish knots reasonably finely (or tell you something else interesting).

- Many new knot invariants have been developed and studied in recent decades.

Symplectic & Contact Homology

- An important recent development in manifold topology is the increased study of *symplectic structures* (non-degenerate 2-forms) on even-dimensional manifolds and the associated *contact structures* (non-integrable hyper-plane fields) on odd-dimensional manifolds. (Highly incomplete list of contributors: Floer, Ozsváth, Szabó, Khovanov, Chekanov, Hutchings, M. Sullivan, Eliashburg, Givental, Hofer.)

- These developments have resulted in novel and powerful homology theories (for example: Heegard-Floer Homology, Symplectic Field Theory, Embedded Contact Homology).

- *Do not worry if you have no idea what any of this means!*

Knot Contact Homology

- Lenhard Ng in 2005 defined the *contact homology* of a knot as the relative contact homology of a Legendrian torus in a 5-dimensional contact manifold that is naturally associated to the knot.

- Knot contact homology as a whole is difficult to compute, but Ng found the degree-zero part, $HC_0(K)$, to be a computable, interesting invariant of the knot K.

- *Do not worry if you have no idea what any of this means! Because...*

The Cord Ring

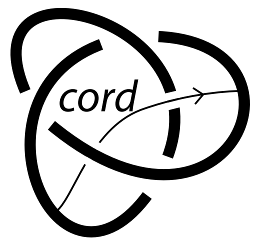
- It turns out that $HC_0(K)$ has a natural, visual, and topologically elementary three-dimensional interpretation, which Ng called the *cord ring*.

- In fact, the cord ring is a stunning example of a knot invariant based on the elementary topology of the knot and some elementary algebra that could easily have been defined in the early 20^{th} century (if not earlier); in fact, it is determined by the fundamental group of the knot complement and the peripheral subgroup of the knot [Ng, Gadgil].

- Except no one thought of it before the development of contact homology!

Cords

- Given a knot K in S^3, define a *cord* to be a continuous (directed) path that intersects the knot only at its endpoints; that is, a continuous map $\gamma : [0,1] \to S^3$ such that $\gamma^{-1}(K) = \{0,1\}$.

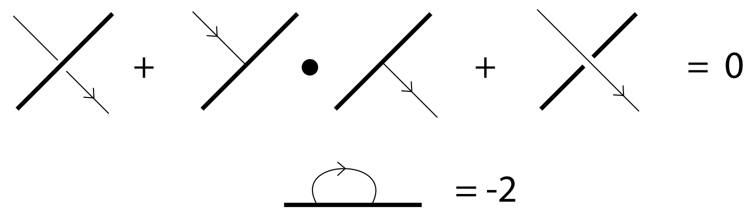


- Two cords are equivalent if they are homotopic through cords (the endpoints of the cord can move but must remain on the knot, and the interior of the cord cannot cross the knot).

The Cord Ring

- Let C_K be the set of all equivalence classes of cords. Let A_K be the tensor algebra over $\mathbb{Z}$ freely generated by C_K.

- Define the following *skein relations*:



- Let I_K be the two-sided ideal generated by all skein relations. The *cord ring* is defined to be A_K/I_K.

- **Theorem [Ng, 2005]:** The cord ring of K is isomorphic to $HC_0(K)$.

Computations

- The skein relations allow one to choose generators for the cord ring that are in a nice position relative to the knot. Taking advantage of this capability, Ng presented some relatively elementary calculation methods.

- Of particular interest, he showed that the cord ring of a two-bridge knot (one that can be embedded in a two-sphere except for two arcs lying outside) is a quotient of the polynomial ring $\mathbb{Z}[x]$, and gave a simple recursive formula for the polynomial that generates the idea of relations.

- However, Ng's methods are not entirely elementary, nor entirely geometric, since they involve representations of the braid group.

Using Surfaces

- If a knot is embedded on some two-sided surface in S^3, possibly with the exception of some "bridges" (arcs with their endpoints on the surface), the cords may all be brought to one side using the skein relations. Except for skein relations involving the bridges (if any), all relations among the generators arise from homotopies on the other side.

- It seemed to me that more elementary and geometric methods were possible, based on the nice position of a two-bridge knot relative to a simple surface (the sphere and, in the case of the simplest two-bridge knots, the $(2, n)$-torus knots, the torus in which they are embedded).

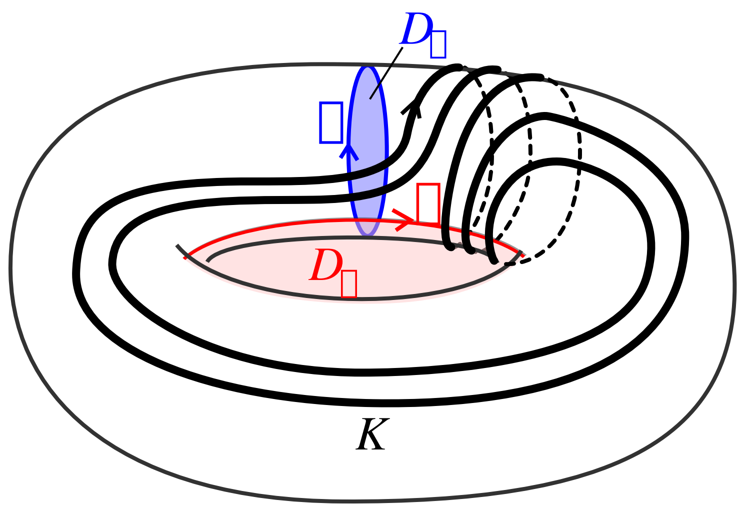
- Susie Wolf (now Susan Brooks) carried out this program for the two-bridge knots as her honors thesis, under my direction, in 2007.

The Cord Ring of a Torus Knot

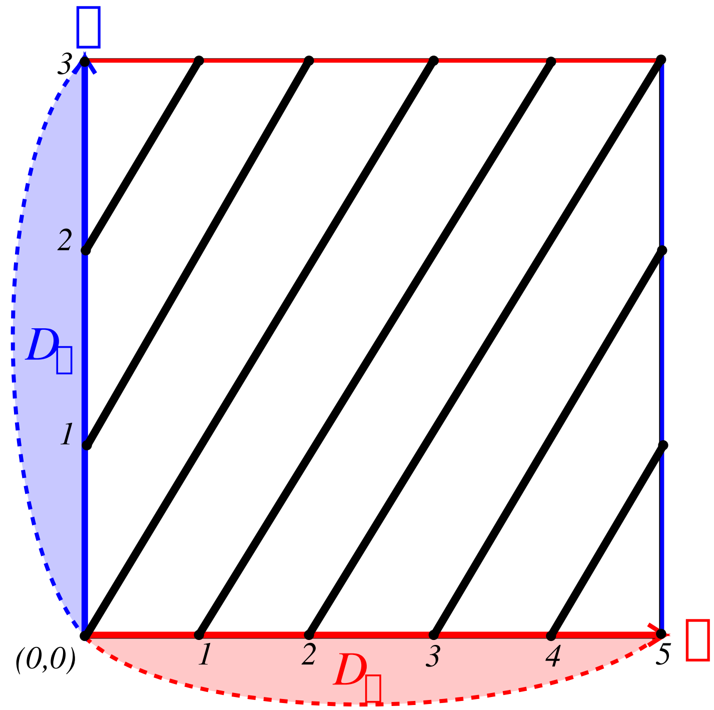
- The natural next step was to calculate the cord ring for general torus knots (knots embedded on an unknotted torus) by this method.

- The absence of any bridges allows for a particularly elegant calculation!

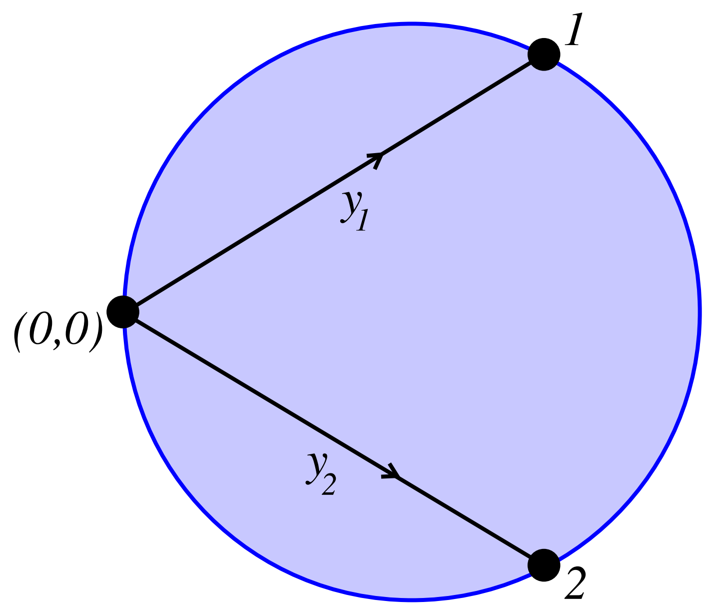
- The fact that a ball is simply connected greatly simplified the homotopy calculations in the two-bridge case. Since the fundamental group of a solid torus is infinite cyclic, these calculations are also very tractable for torus knots.



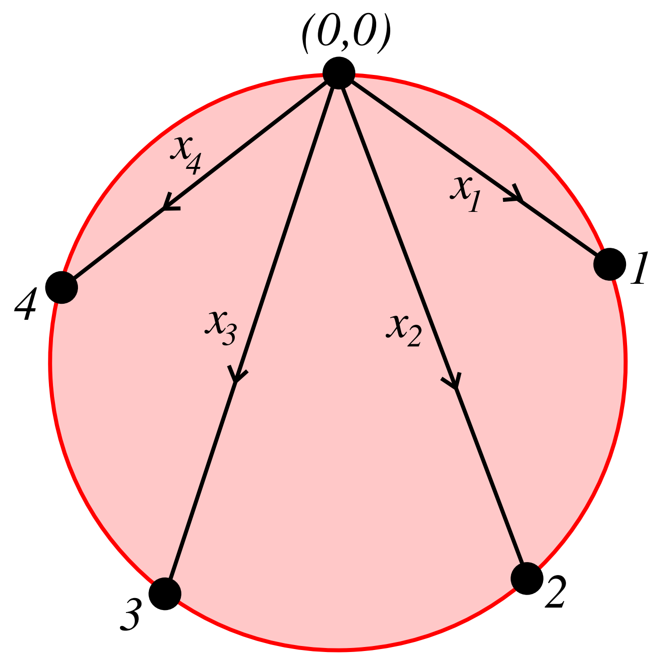
The (oriented) $(2, 3)$ torus knot (trefoil).



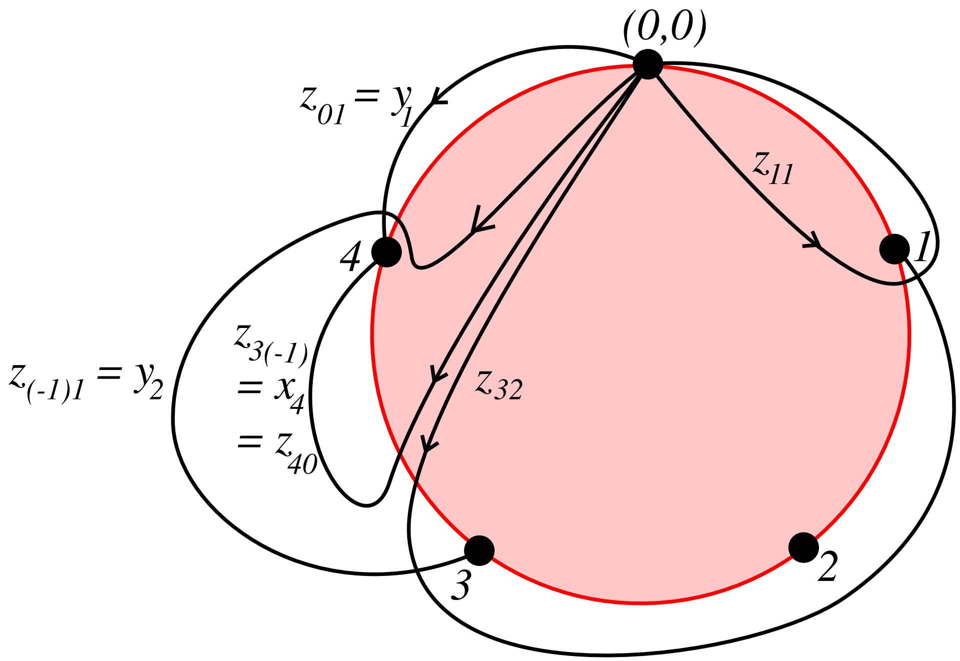
The $(3, 5)$ torus knot on a torus cut into a square.



Representative cords inside the torus.



Representative cords outside the torus.



Representative cords used to calculate the relations. Note that:

$$z_{0j} = y_j \qquad z_{i0} = x_i$$
$$z_{i(-1)} = z_{(i+1)0} = x_{i+1} \qquad z_{(-1)j} = z_{0(j+1)} = y_{j+1}.$$

More generally: $z_{ij} + z_{i0} \cdot z_{0j} + z_{(i-1)(j-1)} = 0.$

The Cords z_{ij} in Terms of the Generators $y_1, y_2, y_3, \ldots, y_{l-1}$

- Keep in mind that $z_{0j} = y_j$ and

$$z_{ij} + z_{i0} \cdot z_{0j} + z_{(i-1)(j-1)} = 0.$$

- Set $y = (y_1, y_2, y_3, \ldots, y_{l-1})$ and $y_0 = -2$.

- For all integers i and j define:
  - $p_{0j}(y) = y_j$ (thus $p_{00}(y) = -2$);
  - $p_{ij}(y) + p_{i0}(y) \cdot p_{0j}(y) + p_{(i-1)(j-1)}(y) = 0$.

The Generating Relations in Terms of the Generators $y_1, y_2, y_3, \ldots, y_{l-1}$

- The generating relations are obtained from the fact that the coefficients of the x_i are defined modulo m.

- There are $l-1$ of them. For example, if m is even and l is odd, we obtain the relations:

  - $p_{(\frac{m}{2})0}(y) - p_{(\frac{-m}{2})0}(y) = 0$
  - $p_{(\frac{m+2}{2})0}(y) - p_{(\frac{2-m}{2})0}(y) = 0$
  - $p_{(\frac{m-2}{2})0}(y) - p_{(\frac{-2-m}{2})0}(y) = 0$

  $\ldots$

  - $p_{\left(\frac{m+2\left(\frac{l-1}{2}\right)}{2}\right)0}(y) - p_{\left(\frac{2\left(\frac{l-1}{2}\right)}{2}-m\right)0}(y) = 0$