MAT 4900: History of Mathematics
Midterm Exam Solutions
(1) (a) Derive a formula for $\sum_{i=1}^{n} i^{2}$ using the division of a triangular prism into three triangular pyramids.
We have seen that $i^{2}=T(i-1)+T(i)$, where $T(i)$ is the $i^{\text {th }}$ triangular number, so the problem is reduced to calculating $\sum_{i=1}^{n} T(i-1)+\sum_{i=1}^{n} T(i)=$ $\sum_{i=1}^{n-1} T(i)+\sum_{i=1}^{n} T(i)$. Let $P(n)=\sum_{i=1}^{n} T(i)$, which we'll call the $n^{\text {th }}$ pyramidal number.
The division of the prism into three pyramids illustrates the equation $P(n)+$ $2 P(n-1)=n T(n)$. The additional recursive equation $P(n)=P(n-$ 1) $+T(n)$ allows us to solve for $P(n)$ and $P(n-1)$, obtaining $P(n-$ $1)=\frac{(n-1) T(n)}{3}=\frac{(n-1)(n)(n+1)}{6}$ (using the known formula $T(n)=\frac{n(n+1)}{2}$ ) and $P(n)=\frac{n(n+1)(n+2)}{6}$. Adding these together we obtain $\sum_{i=1}^{n} i^{2}=P(n-1)+$ $P(n)=\frac{n(n+1)(2 n+1)}{6}$. You may remember this formula from calculus, while having had no idea how anyone every thought of it!
(b) Prove the formula you derived.

This is a simple proof by induction. The case $n=1$ is easily checked: $1=$ $1^{2}=\frac{(1)(2)(3)}{6}$. Assume as inductive hypothesis that $\sum_{i=1}^{n-1} i^{2}=\frac{(n-1)(n)(2(n-1)+1)}{6}=$ $\frac{(n-1)(n)(2 n-1)}{6}$. Then $\sum_{i=1}^{n} i^{2}=\frac{(n-1)(n)(2 n-1)}{6}+n^{2}=\frac{(n-1)(n)(2 n-1)+6 n^{2}}{6}=$ $\frac{n[(n-1)(2 n-1)+6 n]}{6}=\frac{n\left[2 n^{2}-3 n+1+6 n\right]}{6}=\frac{n\left[2 n^{2}+3 n+1\right]}{6}=\frac{n(n+1)(2 n+1)}{6}$.
(2) (a) You may have noticed that in the addition and multiplication tables in bases $b=7$ and $b=12$ the entries in the lower right corner of each, for $(b-1)+(b-1)$ and $(b-1)(b-1)$, have their digits exchanged: $6+6=15_{(7)}$ and $6 \cdot 6=51_{(7)}$; $e+e=1 t_{(12)}$ and $e \cdot e=t 1_{(12)}$ (where $e=11$ and $t=12$ ). Is this true in any base? If so, prove it; if not, give a counterexample.

This relationship is true in any base, as is easily seen by considering the general case. $(b-1)+(b-1)=2 b-2=b+(b-2)$, so its digits, from left to right, are 1 and $b-2 .(b-1)(b-1)=b^{2}-2 b+1=(b-2) b+1$, so its digits, from left to right, are $b-2$ and 1 .
(b) Provide and prove a general formula for the number of digits in the numeral in base $b$ for a number $n$, in terms of $b$ and $n$.

A number $n$ in base $b$ has $N$ digits if $b^{N-1} \leq n<b^{N}$; equivalently, $N-1 \leq$ $\log _{b} n<N$. Thus, $N$ is the least integer strictly greater than $\log _{b} n$. This is most succinctly written as $N=\left\lfloor\log _{b} n\right\rfloor+1$, where $\lfloor\quad\rfloor$ denotes the integer part.
(c) Provide a general description of the regular numbers and their reciprocals as radix fractions in base 2 and in base 4 .

A number is regular in base 2 if it is a power of 2 . The reciprocal of $2^{n}$ has radix fraction $.0 \cdots 01$, with $n-1$ zeros. The regular numbers in base 4 are also the powers of two, but the radix fraction numerals for their reciprocals are different. There are two cases: either the regular number is a power of 4, say $4^{n}$, in which case its reciprocal has the form $.0 \cdots 01$, with $n-1$ zeros, or it is twice a power of 4 , say $2 \cdot 4^{n}$, in which case its reciprocal has the form $.0 \cdots 02$, with $n-1$ zeros.
(3) (a) Derive and justify, both geometrically and analytically, a formula for the best linear approximation to $\sqrt[3]{a^{3}+h}$. (In other words, derive and justify the correct coefficient $k$ in the formula $\sqrt[3]{a^{3}+h}=a+k h$.)
For the geometric perspective, it would perhaps be helpful to review the case of square roots, for which our picture is two dimensional. In Figure 1 , $\sqrt{a^{2}+h}=a+\Delta$, where $h$ and $\Delta$ are positive in the figure at left and negative in the figure at right. In both figures, the gray area is the magnitude of $h$. In the figure at left, for which $a$ underestimates the square root, each darker gray rectangle has area slightly less than $\frac{h}{2}$ (the difference being half the small square in the corner), so $\Delta$ is slightly less than $\frac{h}{2 a}$. Thus, $a+\frac{h}{2 a}$ is the best linear approximation and is slightly too large. (We know it is the best linear approximation because the difference from the exact answer is proportional to $\Delta^{2}$.) In the figure at right, for which $a$ overestimates the square root, the darker gray rectangles, which now overlap in the corner, each have area slightly more than $\frac{h}{2}$ (with the difference again being half the square in the corner), so $\Delta$ is slightly more than $\frac{h}{2 a}$. Thus, in adding $\frac{h}{2 a}$ we are subtracting less than $\Delta$, so the best linear approximation, $a+\frac{h}{2 a}$, remains an overestimate.


Figure 1
To find the best linear approximation to $\sqrt[3]{a^{3}+h}$ we simply need a threedimensional version of this picture. In Figure 2 I present the case that $h$ and $\Delta$ are positive, since it is easier to visualize in three dimensions. I have not shaded the entire difference in volumes, $h$, since the smaller cube is hidden, and hence the shading added nothing helpful to the picture. Each of the shaded blocks is slightly less than $\frac{h}{3}$ (the difference being the three rectangular solids with square cross section along the edges of the smaller cube, which have volume proportional to $\Delta^{2}$, together with the cube in the corner, which of course has volume $\Delta^{3}$ ). Thus, $\Delta$ is slightly less than $\frac{h}{3 a^{2}}$, and the best linear approximation to $\sqrt[3]{a^{3}+h}$, which is once again an overestimate, is $a+\frac{h}{3 a^{2}}$.
To derive this result analytically, we determine the linear Taylor polynomial for the function $f(h)=\sqrt[3]{a^{3}+h}$, centered at $h=0 . f^{\prime}(h)=\frac{1}{3}\left(a^{3}+h\right)^{-\frac{2}{3}}$; hence, $f^{\prime}(0)=\frac{1}{3 a^{2}}$, yielding the same result as we obtained geometrically above.
(b) Compare your result to the the approximation $x_{1}$ given by Newton's method applied to $f(x)=x^{3}-\left(a^{3}+h\right)$ with $x_{0}=a$.


Figure 2
Applying the formula $x_{1}=x_{0}-\frac{f\left(x_{0}\right)}{f^{\prime}\left(x_{0}\right)}$ to $x_{0}=a$ once again yields $x_{1}=a+\frac{h}{3 a^{2}}$. Furthermore, since the graph $y=f(x)$ is concave upward for $x>0$, we can see by following the tangent lines that $x_{1}$, as well as every subsequent approximation, is an overestimate.
(4) Solve the quadratic equation $x^{2}-2 x-2=0$ geometrically, using the Pythagorean method. That is, using only straightedge and compass, construct segments whose lengths are the absolute values of the roots of this equation. Clearly show your construction, and fully document and justify it. One of the roots is negative. Explain which one it is and how you know, without referring to the quadratic formula for the roots.
$x^{2}-2 x-2=0 \Leftrightarrow x(x-2)=2$. We easily construct $\sqrt{2}$ as the hypotenuse of a right triangle with legs of length 1 . (In general we would use the semi-circle construction, but it is not necessary if the number we are finding the square root of is a sum of two squares; another alternative is the spiral of Theodorus, discussed in Exercise 3.15. It clearly does not matter how one constructs the square root of the constant.) Now, turning to Figure 18 on page 87 , we have $A B=2, A Q=x$, $B Q=x-2$, and $A P=1$. One solution is $x=A Q$, of course, and to construct the point $Q$ we use that $(P Q)^{2}=(A Q)(B Q)+(P B)^{2}=2+1=3$. Thus, we construct $P Q$ as the hypotenuse of a right triangle with legs 1 and $\sqrt{2}$, and we construct $x=A Q$ by adding 1 to $P Q$, obtaining $x=1+\sqrt{3}$. The construction is shown in Figure 3.

To determine the other root, we do a little algebra: setting $y=2-x$, so that $x=2-y$ and $x-2=-y$, we obtain $2=x(x-2)=(2-y)(-y)=y(y-2)$, so $y=-B Q$ is the other solution.
(5) (a) Prove that if $p$ and $q$ are distinct prime numbers, then $\sqrt{p}$ and $\sqrt{q}$ are incommensurable.
Suppose by way of contradiction that, for some positive integers $k$ and $l$, $k \sqrt{p}=l \sqrt{q}$. Then $k^{2} p=l^{2} q$. The prime factor $p$ must occur an odd number of times in $k^{2} p$ and an even number of times in $l^{2} q$, which is impossible. (Similarly, the prime factor $q$ must occur an even number of times in $k^{2} p$ and an odd number of times in $l^{2} q$, but this additional contradiction is not needed for the proof.)


Figure 3
(b) Need it be true that $\sqrt[3]{p}$ and $\sqrt[3]{q}$ are incommensurable? If so, prove it. If not, give a counterexample.
Yes. Similar reasoning applies, but now we are considering how the number of factors differs from a multiple of 3 . Suppose by way of contradiction that, for some positive integers $k$ and $l, k \sqrt[3]{p}=l \sqrt[3]{q}$. Then $k^{3} p=l^{3} q$. The number of times $p$ occurs as a factor of $k^{3} p$ is one more than a multiple of 3 , whereas the number of times $p$ occurs as a factor of $l^{3} q$ is exactly a multiple of 3 (possibly 0 ). Since the remainder when dividing by 3 is uniquely determined, these numbers cannot be equal, and we have a contradiction. Or, more simply and specifically put, a number cannot be both a multiple of 3 and not a multiple of 3 . (As before, counting the factors of $q$ also leads to a contradiction.)
(c) What can we say in general about $\sqrt[n]{p}$ and $\sqrt[n]{q}$, where $n$ is a positive integer greater than 1? Prove your answer is correct!
In general, $\sqrt[n]{p}$ and $\sqrt[n]{q}$ are incommensurable. The remainder when a whole number is divided by $n$ is uniquely determined as one of $\{0,1,2, \ldots, n-1\}$. Suppose by way of contradiction that, for some positive integers $k$ and $l$, $k \sqrt[n]{p}=l \sqrt[n]{q}$. Then $k^{n} p=l^{n} q$. The number of times $p$ occurs as a factor of $k^{n} p$ is one more than a multiple of $n$, whereas the number of times $p$ occurs as a factor of $l^{n} q$ is exactly a multiple of $n$, so we once more obtain a contradiction.

