**MAT 4900: History of Math.** **Name:** ____________________________

**Final Exam**

**Due by 5 p.m. on Friday, December 15, 2017.**

*Note: Electronic signatures and submissions are acceptable and encouraged, but not required; however, I will be out of town after Wednesday, December 13, (leaving sometime that afternoon) so if you wish to hand it in after that, you will need to do so electronically.*

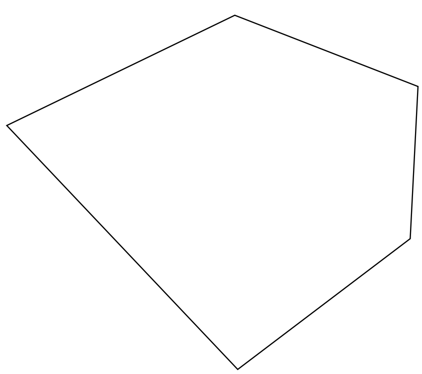
You are expected to work on this exam alone and to refrain from talking about the exam to anyone except the professor until the time and date when it is due. You may use your own notes and any published materials that you like. (Cite sources appropriately.)

Your signature below attests to a pledge that you have done the exam according to the above instructions. Your exam *must* be signed!

**Signature:**____________________________________

*Solutions must be typeset.*

1. Using only straightedge and compass, construct (by hand, not using a computer) a triangle with the same area as the pentagon below. (You need not, however, use a *collapsible* compass, as the ancient Greeks would have required!)



2. As Nathaniel noted, Sanskrit prosody may have well been the first occurrence of the sequence we now associate with the name Fibonacci. Let's see why! It is not difficult. The problem is to partition a line of $n$ syllables into groups of one or two syllables each. (Similar considerations arise in musical rhythm as well, usually with groups of two or three beats.)

   Let $P_n$ denote the number of distinct such partitions. You may find it helpful to visualize them with dots and dividing lines. For example, the figure illustrates that $P_4 = 5$.

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   Observe that the first group may either contain one syllable (or beat or whatever), in which case there remain $n-1$ syllables to be partitioned, or the first group may contain two syllables, in which case there remain $n-2$ syllables to be partitioned. Using this observation, show that $P_n = P_{n-1} + P_{n-2}$. Since $P_1 = 1$ and $P_2 = 2$, it follows that $P_n$ is the $(n+1)^{\text{st}}$ Fibonacci number, if we count 1 as the first, 1 as the second, 2 as the third, and so on.

3. The *golden ratio* may be described as follows: For simplicity, let the shorter side of a rectangle have length 1 and the longer side have length $\rho$. If the ratio between the sides is such that, when a square of side one is cut from the rectangle, the smaller rectangle remaining has sides in the same ratio as the larger one, then $\rho$ is the *golden ratio.* A rectangle with sides in this ratio is called a *golden rectangle.*

   (a) Show that this geometric description leads to the equation $\rho^2 - \rho - 1 = 0$.

   (b) Solve this equation, and construct a golden rectangle with straightedge and compass.

   (c) Show also that $\rho = 1 + \frac{1}{\rho}$. (This equation will be useful below.)

4. An interesting and beautiful connection between the Fibonacci sequence and classical mathematics is that the limit of the ratio of successive terms in the Fibonacci sequence is the golden ratio, as Nathaniel also discussed. For this purpose it is best to let the first term of the sequence be 1, so that this ratio is always defined. Thus, define the Fibonacci sequence $(a_n)_{n=0}^{\infty}$ recursively as follows: $a_0 = a_1 = 1$ and, for $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$. For $n = 0, 1, 2, 3, \ldots$, define $b_n = \frac{a_{n+1}}{a_n}$. Thus, $(b_n)_{n=0}^{\infty}$ is the sequence of ratios of successive terms of the Fibonacci sequence.

   Nathaniel showed the following assertions in his talk, so I am just asking you to recall the arguments here, which should be easy.

   (a) Prove that, for $n \geq 1$, $b_n = 1 + \frac{1}{b_{n-1}}$.

   (b) Let $b = \lim_{n\to\infty} b_n$, assuming this limit exists, which is a major assumption! Noting that, clearly, $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b_{n-1} = b$ (since $b_{n-1}$ just follows one step behind), prove that $b = \rho$. (You may use the properties of limits proven in calculus; you don't need to go back to epsilons and deltas!)

I think the argument above is the most intuitive and straightforward way to understand why $\lim_{n\to\infty} b_n = \rho$. However, not every sequence has a limit, of course, so this argument is not justified without a proof that $\lim_{n\to\infty} b_n$ actually exists. It only shows that *if* the sequence $(b_n)$ has a limit as $n \to \infty$, that limit must be $\rho$.

(c) As an illustration of the importance of this point, show that, although the solution to the equation $x = 2x - 1$ is clearly $x = 1$, the sequence defined recursively by $a_0 = 1$ and $a_{n+1} = 2a_n - 1$ does not converge to 1.

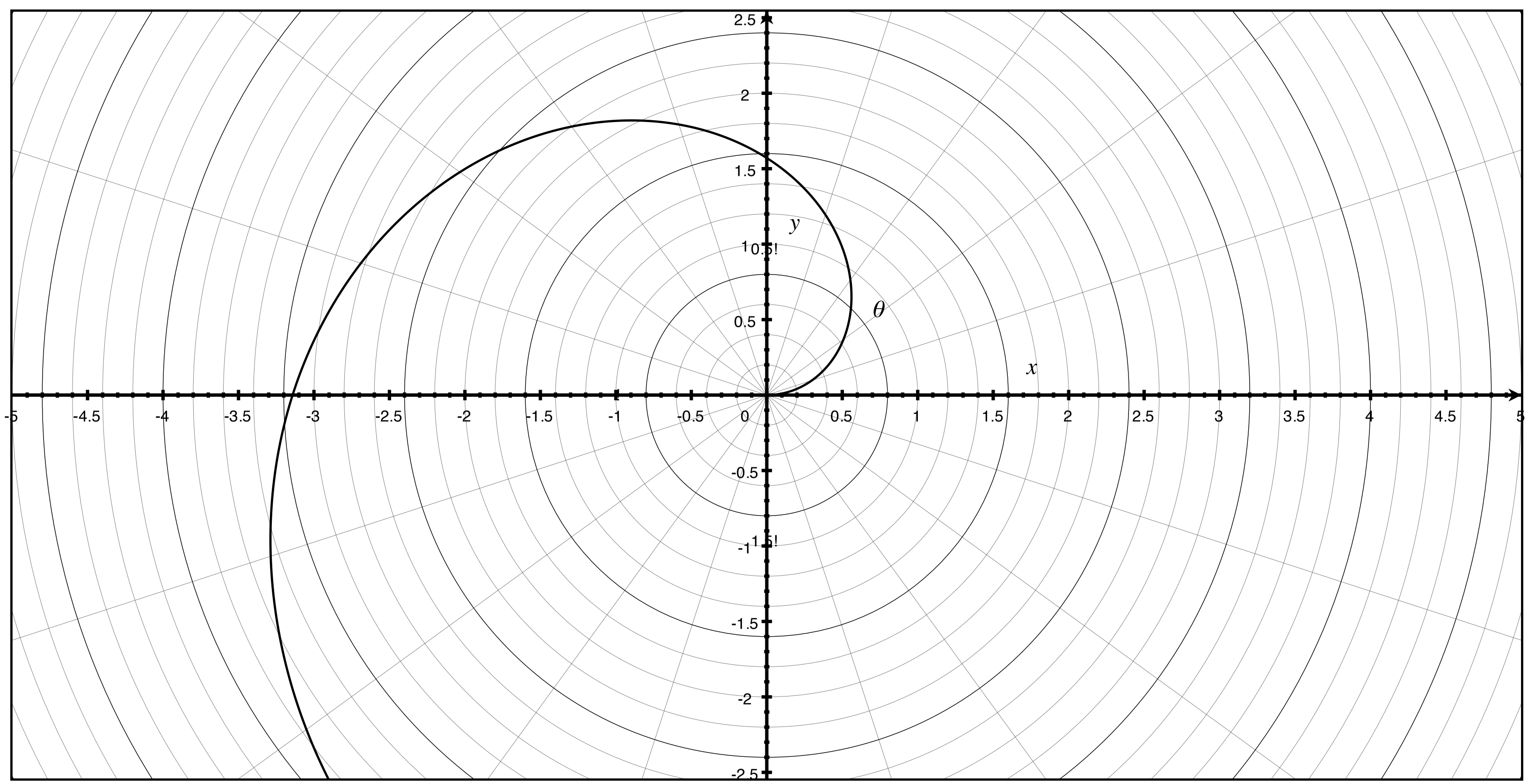
(d) To fill the gap noted above, we will now show directly that $\lim_{n\to\infty} b_n = \rho$ by showing that $\lim_{n\to\infty} b_n - \rho = 0$. First, it should be obvious that all terms in the Fibonacci sequence are positive. As a consequence it is clear that, for all $n \geq 0$, $b_n \geq 1$. (Technically, one should prove those things by induction, but we'll let it go!) We now proceed in three small steps:

i. Using the fact that $\rho = \frac{1}{1+\rho}$, prove that for all $n \geq 1$, $|b_n - \rho| \leq \frac{1}{\rho}|b_{n-1} - \rho|$. Hint: $|b_n - \rho| = |(1 + \frac{1}{b_{n-1}}) - (1 + \frac{1}{\rho})| = |\frac{\rho - b_{n-1}}{\rho b_{n-1}}|$.

ii. Prove that for all $n \geq 1$, $|b_n - \rho| \leq \frac{1}{\rho^n}|b_0 - \rho| = \frac{1}{\rho^n}|1 - \rho|$. (Hint: induction on n!)

iii. Conclude that $\lim_{n\to\infty} b_n - \rho = 0$.

5. The figure on the following page is a spiral of Archimedes. Using straightedge and compass, construct a $60°$ angle and, with the aid of the spiral, trisect it.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