Math 4810J: Solutions to The Practice Problems

I. 1. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \sum_{i=0}^{\infty}\frac{1}{3^i} = \frac{3}{2}$, as shown in class.

Note that we can derive a general formula for $\sum_{i=0}^{\infty} r^i$ if $|r|<1$:

$$
\begin{aligned}
\text{Let } S_n &= 1+r+r^2+\cdots+r^n \\
\cancel{\text{then}} - rS_n &= \quad\; r+r^2+r^3+\cdots+r^n+r^{n+1} \\
\hline
(1-r)S_n &= 1-r^{n+1}.
\end{aligned}
$$

If $|r|<1$, then $r^{n+1}\to 0$ as $n\to\infty$, so $(1-r)S = 1 \iff S = \frac{1}{1-r}$.

The problem above has $r=\frac{1}{3}$, so $S = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$.

2. $\sum_{i=0}^{\infty}\frac{2}{3^i} = 2\sum_{i=0}^{\infty}\frac{1}{3^i} = 2\left(\frac{3}{2}\right) = 3.$

3. $3+\sum_{i=1}^{\infty}\frac{2}{3^i} = 3+\frac{2}{3}\sum_{i=0}^{\infty}\frac{1}{3^i} = 3+\frac{2}{3}\left(\frac{3}{2}\right) = 3+1 = 4.$ This is another way to do it, slightly different from what we did in class.

lower powers by 1

4. $\sum_{i=0}^{\infty}\left(\frac{2}{3}\right)^i = \frac{1}{1-\frac{2}{3}} = \frac{1}{\frac{1}{3}} = 3.$ You can use the formula above with $r=\frac{2}{3}$.

II. 1. $\sqrt{8} = 2\sqrt{2}$. The computations are easier for $\sqrt{2}$:

Let $x_0 = \frac{3}{2}$. Then $x_1 = \dfrac{\frac{3}{2}+\frac{2}{3/2}}{2} = \dfrac{\frac{3}{2}+\frac{4}{3}}{2} = \frac{9+8}{12} = \frac{17}{12}$

$$x_2 = \frac{\frac{17}{12}+\frac{2}{17/12}}{2} = \frac{\frac{17}{12}+\frac{24}{17}}{2} = \frac{577}{408} \approx \sqrt{2}, \text{ so } \sqrt{8} \approx 2x_2 = \frac{577}{204}.$$

Alternatively, you can compute an estimate of $\sqrt{8}$ directly. A good first approximation is 3, since $3^2=9$.

Let $x_0 = 3$. $x_1 = \dfrac{3+\frac{8}{3}}{2} = \dfrac{17}{6}$. (Note that this is twice what we got the other way!)

$$x_2 = \frac{\frac{17}{6}+\frac{8}{17/6}}{2} = \frac{\frac{17}{6}+\frac{48}{17}}{2} = \frac{577}{204}$$

Same answer. On second thought, maybe that was easier than the first method!

2. For $\sqrt{17}$, start w/ $x_0=4$.

$$x_1 = \frac{4+\frac{17}{4}}{2} = \frac{33}{8}. \quad x_2 = \frac{\frac{33}{8}+\frac{17}{33/8}}{2} = \frac{\frac{33}{8}+\frac{136}{33}}{2} = \frac{2177}{528} \approx \sqrt{17}$$

III. 1. Let $s_n = \sum_{i=1}^{n} 2^i = 2+2^2+\cdots+2^n$. Then

$$s_n = 2s_n - s_n = \begin{cases} 2^2+2^3+\cdots+2^{n+1} \\ -2-2^2-2^3-\cdots-2^n \end{cases}$$
$$2^{n+1}-2 = 2(2^n-1).$$

Here's a nice general method: the trick we used above works in general for partial sums (but the limit as $n\to\infty$ is infinite of $|r|\geq 1$ so does not exist): $s_n = \dfrac{1-r^{n+1}}{1-r}$. But remember the sum must start with $1=r^0$. To make the given sum fit the formula, take out a factor of 2: $\sum_{i=1}^{n} 2^i = 2\sum_{i=0}^{n-1} 2^i = 2\left(\dfrac{1-2^n}{1-2}\right)$
$= 2\left(\dfrac{2^n-1}{2-1}\right) = 2(2^n-1)$.

2. Using the method above, $\sum_{i=1}^{n} 3^i = 3\sum_{i=0}^{n-1} 3^i = 3\left(\dfrac{1-3^n}{1-3}\right) = 3\left(\dfrac{3^n-1}{3-1}\right)$
$= 3\left(\dfrac{3^n-1}{2}\right) = \dfrac{3}{2}(3^n-1)$.

3. $\sum_{i=1}^{n} \dfrac{1}{2^i} = \dfrac{1}{2}\sum_{i=0}^{n-1}\dfrac{1}{2^i} = \dfrac{1}{2}\left(\dfrac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}\right) = \dfrac{1}{2}\left(\dfrac{1-(\frac{1}{2})^n}{\frac{1}{2}}\right) = 1-\left(\dfrac{1}{2}\right)^n$.

4. $\sum_{j=1}^{n} \dfrac{1}{3^j} = \dfrac{1}{3}\sum_{j=0}^{n-1}\dfrac{1}{3^j} = \dfrac{1}{3}\left(\dfrac{1-(\frac{1}{3})^n}{1-\frac{1}{3}}\right) = \dfrac{1}{3}\left(\dfrac{1-(\frac{1}{3})^n}{\frac{2}{3}}\right) = \dfrac{1}{\cancel{3}}\cdot\dfrac{\cancel{3}}{2}\left(1-\left(\dfrac{1}{3}\right)^n\right) = \dfrac{1}{2}\left(1-\left(\dfrac{1}{3}\right)^n\right)$.

Note that your answers might be different expressions equal to these if you used a different method. $n(2n+1)$.

5. $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ 6. $\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}$ 7. $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$ 8. $\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2} =$

IV. 1. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5}{2^i} = \lim_{n\to\infty} 5 \sum_{i=1}^{n} \frac{1}{2}^i = \lim_{n\to\infty} 5\left(1-\left(\frac{1}{2}\right)^n\right) = 5.$

2. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{5t^2}{2^i} = \lim_{n\to\infty} 5t^2\left(1-\frac{1}{2^n}\right) = 5t^2.$

3. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2^i}{2^n} = \lim_{n\to\infty} \frac{1}{2^n}\sum_{i=1}^{n} 2^i = \lim_{n\to\infty} \frac{1}{2^n}\left(2(2^n-1)\right) = \lim_{n\to\infty} \frac{2\cdot\cancel{2^n}}{\cancel{2^n}} - \frac{2}{2^n}$

$= \lim_{n\to\infty} 2 - \frac{2}{2^n} = 2.$ $\to 0$

4. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{t^2 2^i}{2^n} = \lim_{n\to\infty} \frac{t^2}{2^n}\sum_{i=1}^{n} 2^i = 2t^2.$

5. $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{5}{n^2} j = \lim_{n\to\infty} \frac{5}{n^2}\sum_{j=1}^{n} j = \lim_{n\to\infty} \frac{5}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty} \frac{5}{2}\left(\frac{n^2+n}{n^2}\right) =$

$\lim_{n\to\infty} \frac{5}{2}\left(1-\frac{1}{n}\right) = \frac{5}{2}$

6. $\lim_{n\to\infty} \sum_{j=1}^{n} \frac{5jx}{n^2} = \lim_{n\to\infty} \frac{5x}{n^2}\sum_{j=1}^{n} j = \frac{5x}{2}$

7. $\lim_{n\to\infty} \sum_{j=1}^{2n} \frac{5jx^2}{n^2} = \lim_{n\to\infty} \frac{5x^2}{n^2}\sum_{j=1}^{2n} j = \lim_{n\to\infty} \frac{5x^2}{n^2}\left(n(2n+1)\right) =$

$\lim_{n\to\infty} 5x^2\left[\frac{2n^2+n}{n^2}\right] = \lim_{n\to\infty} 5x^2\left[2+\frac{1}{n}\right] = 10x^2.$

V. Since all of these are handled the same way, I'll just do the ones we did not do in class.

1. $f'(x) = \lim_{\Delta x\to 0} \frac{5-5}{\Delta x} = \lim_{\Delta x\to 0} \frac{0}{\Delta x} = \lim_{\Delta x\to 0} 0 = 0.$

2. $f'(x) = \lim_{\Delta x\to 0} \frac{[3(x+\Delta x)^2+5]-[3x^2+5]}{\Delta x} = \lim_{\Delta x\to 0} \frac{3(x^2+2x\Delta x+\Delta x^2)\cancel{+5}-3x^2\cancel{-5}}{\Delta x} =$

$\lim_{\Delta x\to 0} \frac{\cancel{3x^2}+6x\Delta x+3\Delta x^2-\cancel{3x^2}}{\Delta x} = \lim_{\Delta x\to 0} 6x+3\Delta x = 6x.$

3. $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{[7(x+\Delta x)^3+2]-[7x^3+2]}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{7(x^3+3x^2\Delta x+3x\Delta x^2+\Delta x^3)+\cancel{2}-7x^3-\cancel{2}}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{\cancel{7x^3}+21x^2\Delta x+21x\Delta x^2+\Delta x^3-\cancel{7x^3}}{\Delta x} = \lim\limits_{\Delta x \to 0} 21x^2+21x\Delta x+\Delta x^2 = 21x^2.$

4. $f'(x) = \lim\limits_{\Delta x \to 0} \dfrac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x}\left(\dfrac{x-(x+\Delta x)}{x(x+\Delta x)}\right) = \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x}\left(\dfrac{x-x-\Delta x}{x(x+\Delta x)}\right)$

$= \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x}\left(\dfrac{-\Delta x}{x(x+\Delta x)}\right) = \lim\limits_{\Delta x \to 0} \dfrac{-1}{x(x+\Delta x)} = \dfrac{-1}{x^2}.$

VI. 1. At $t=10$, $v = g(10) \approx 98$ meters per second

2. At $t=20$, $v = 20g \approx 196$ meters/second.

3. $v = gt$.

4. At time $\tau = \frac{it}{n}$, $v = g\left(\frac{it}{n}\right) \approx 9.8\left(\frac{it}{n}\right)$.

5. We know that in t seconds the object falls $s = \frac{1}{2}gt^2$ meters. Letting $h = H(t)$ be height above the ground, we have $H(10) \approx$
$4410 - \frac{1}{2}(9.8)(10)^2 = 4410 - 490 = 3920$ meters above the ground.

6. $H(20) \approx 4410 - 4.9(20)^2 = 4410 - 1960 = 2450$ meters above the ground.

7. $h = H(t) = 4410 - \frac{1}{2}gt^2 \approx 4410 - 4.9t^2$

8. $h = 0$ when $4.9t^2 = 4410$. This occurs when $t^2 = \frac{4410}{4.9} = 900$,
so $t = 30$ seconds.