MAT 4810: Calculus for Elementary and Middle School Teachers

Computing Derivatives & Integrals Part III: Optimization

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Optimization

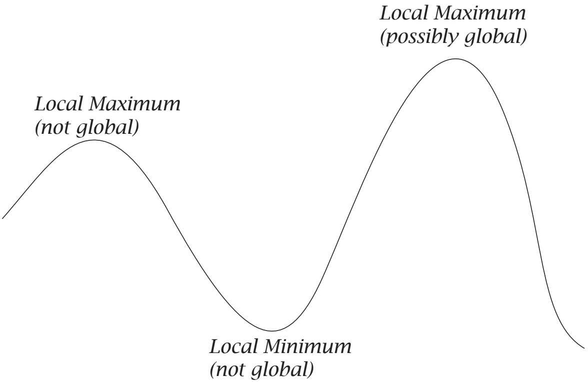
Calculus can often help us find the "best" way do something.

- We must determine how some desired outcome depends on other factors. For example, the volume of a box depends on its dimensions.
- There are generally some constraints. For example, we may have a specified amount of material for the box, and it might have to have a certain shape, such as having a square bottom.
- We must decide what "best" means in quantitative terms. For example, we most likely would like to maximize the volume of the box, given the amount of material available.
- The derivative allows us to solve for which inputs give maximal or minimal outputs. Let's see how!

Local Maximum and Minimum Values

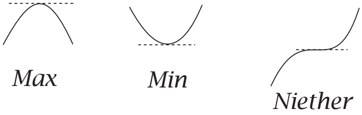
- A function f has a *local maximum* at an input x if $f(x)$ is greater than all nearby output values: the graph of $y = f(x)$ reaches a "peak."
- The output value of f at a local maximum need not be the largest output value, or *global* maximum: there may be several peaks, some perhaps higher than others.
- A function f has a *local minimum* at an input x if $f(x)$ is less than all nearby output values.
- As in the case of a local maximum, the value of the function at a local minimum need not be its smallest value overall, or global minimum.
- "Nearby" means in some surrounding interval, which can be as small as necessary to avoid other peaks and valleys.

Graphical View of Maxima & Minima



What the Derivative Tells Us About Local Maxima and Minima

- When a function reaches a maximum or minimum value, it is neither increasing or decreasing; hence, its derivative (if defined) will be zero. (An example where the derivative is not defined is the minimum of $f(x) = |x|$ at $x = 0$.)
- Suppose $f'(c) = 0$.
  - If $f'(x) > 0$ for nearby inputs $x < c$, and $f'(x) < 0$ for nearby inputs $x > c$, then $f(c)$ is a local maximum.
  - If $f'(x) < 0$ for nearby inputs $x < c$, and $f'(x) > 0$ for nearby inputs $x > c$, then $f(c)$ is a local minimum.
  - It is possible that $f(c)$ is neither a local maximum nor a local minimum.



An Example Where the Minimum is Best

- More is not always better!
- As another example, in constructing a cylindrical can to hold a certain volume, we want to minimize the amount of metal needed; hence, we seek to minimize the surface area.
- In general, the surface area depends on both the height and the radius of the can.
- But because of our constraint on the volume, the height depends on the radius. (A fatter can will be shorter, in order to hold the same amount.) So the surface area is a function of the radius alone.
- By taking the derivative of the surface area, finding the possible local maxima and minima, and comparing (if necessary) their values, we can find the can with the least surface area.

Let's try some examples in detail, starting with easy ones and working up to harder ones!