MAT 4810J: Calculus for Elementary and Middle School Teachers

Modeling an Epidemic

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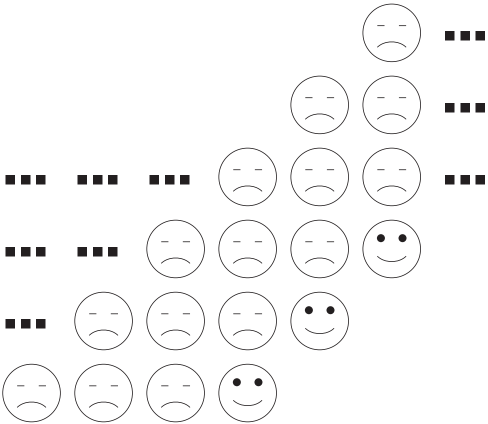
Understanding Recovery

- If the disease lasts 4 days, then one person out of four recovers each day.



The Units of the Recovery Coefficient

- That is:

$$\frac{\frac{1 \text{ person}}{4 \text{ persons}}}{\text{day}} = \frac{.25}{\text{day}}$$

- So the units are

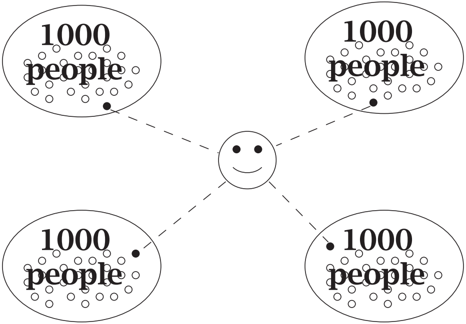
$$\frac{\frac{\text{persons}}{\text{persons}}}{\text{days}} = \frac{1}{\text{days}}$$

- When we multiply by I, the number of infected persons, we get .25 persons per day, with units of

$$\frac{\text{persons}}{\text{days}}$$

Breaking Down the Transmission Coefficient

- Suppose that an average person comes into close contact with one out of every thousand people in the given population each day.



Breaking Down the Transmission Coefficient

- That is, one contact per thousand persons per person per day, or:

$$\frac{\frac{\frac{1\text{ contact}}{1000\text{ persons}}}{\text{person}}}{\text{day}} = .001\frac{\text{contacts}}{\text{persons}^2\cdot\text{days}}$$

- The units (so far) are:

$$\frac{\text{contacts}}{\text{persons}^2\cdot\text{days}}$$

- Alternatively, if we wish to think of contacts as ill people (who are in contact with susceptible people), we could view the units as

$$\frac{\text{(ill) persons}}{\text{((ill) persons)((susceptible) persons)}\cdot\text{days}} = \frac{1}{\text{persons}\cdot\text{days}}$$

Breaking Down the Transmission Coefficient

- Not every contact results in a transmission of the illness.
- Suppose one contact in ten results in transmission:

$$\frac{1 \text{ transmission}}{10 \text{ contacts}} = .1 \frac{\text{transmissions}}{\text{contact}}$$

- Thus there are:

$$\left(.1 \frac{\text{transmissions}}{\text{contact}}\right)\left(.001 \frac{\text{contacts}}{\text{persons}^2 \cdot \text{days}}\right)$$
$$= .0001 \frac{\text{transmissions}}{\text{persons}^2 \cdot \text{days}}$$

- In other words, .0001 transmission per (susceptible) person per (infected) person per day, which came from 1 transmission per 10 (susceptible) persons per 1000 (infected) persons per day.

The Transmission Rate

- As we have just seen, the transmission coefficient has units of:

$$\frac{\text{transmissions}}{\text{persons}^2 \cdot \text{days}}$$

- Thus, when we multiply by S, the number of susceptible people, and by I, the number of infected people, we obtain a a transmission rate in units of transmissions per day:

$$.0001SI \, \frac{\text{transmissions}}{\text{days}}$$

Interpreting the Transmission Rate

- Of course, a "transmission" just means a person who moves from the susceptible population to the ill population. That is, a "transmission" is just a particular kind of person - a person who has caught the illness.
- So we get a flow from the susceptible population to the ill population in units of persons per day:

$$.0001SI \, \frac{\text{persons}}{\text{days}}$$

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Adding Another Flow: Losing Immunity

- If people lose immunity after a period of time, we see another flow: from the recovered population into back into the susceptible population. (The label "recovered" implies immunity.)
- As with the recovery rate, the rate of this flow depends in the length of time a person remains immune.
- For example, if immunity lasts for 100 days, then one person per hundred (recovered) people will become susceptible per day.
- Thus the immunity loss coefficient would be .01 persons per person per day.
- Multiplying by the number of recovered (and immune) people gives a flow rate in persons per day.

The Equations with Temporary Immunity

- Under conditions of temporary immunity, the rate equations assume a symmetrical form, since each population has a flow in and a flow out.
- Using the parameters a, b, and c to represent the coefficients of transmission, recovery, and immunity loss, respectively, the equations are:

$$S' = cR - aSI$$

$$I' = aSI - bI$$

$$R' = bI - cR$$

- Note how each flow rate appears in the rate of change for two populations, negatively for the population it flows out of and positively for the population it flows into.

What if People Die?

- Suppose 10% of those who get ill will die and the average number of days from onset to death is two.
- Then what percentage of ill people die per day?
- In general, let us denote the percentage of ill people who die per day, to which we might refer as the *death coefficient*, by the parameter d.
- This percentage might depend on various factors: the virulence of the particular strain of the disease, the health and nutrition of the population, the quality of medical care available, and so forth.)
- What would the system diagram look like?
- Now we need a fourth equation, for D', where D is the number of people who are dead as a result of the illness. What is it?

The Units of the Death Coefficient

- As with the transmission coefficient, the death coefficient depends on two factors.
- Just as not every contact results in transmission, not every illness results in death. Taking the 10% figure as an example, we have:

$$\frac{1 \text{ person (who dies)}}{10 \text{ persons (who are ill)}} = .1$$

- For every case of the illness that does result in death, it takes a certain average number of days for the ill person to die. Taking the figure of 2 days as an example, we have:

$$(.1)\frac{\frac{1 \text{ person (who dies that day)}}{2 \text{ persons (who ultimately die)}}}{\text{day}} = .05\frac{1}{\text{day}}$$

The Units of the Death Coefficient

- In other words, 10% of those who are ill will die, and each day half of those who will die do die on that day, yielding a death coefficient of 5% (of the ill population) per day:

$$d = .05 \frac{1}{\text{day}}$$

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- Multiplying by the number of ill people, I, we obtain a death rate of .05 persons per day:

$$D' = .05I \frac{\text{persons}}{\text{day}}$$

Incorporating Death into the Model

- In general, what is D' in terms of d and I?
- How does d affect the recovery rate?
- The dead population has a flow in, but of course it does not have a flow out. Although there are situations where immunity can be lost, creating a flow out of the recovered population, death is always ... well ... permanent.
- How would you incorporate the death rate into the rate equations that model the epidemic?

A More Complete Model Template

- You should now have the following set of equations in terms of four population variables, S, I, R, and D, and five rate parameters, a, b, c, d_1 and d_2, where d_1 is the percentage of infected people who die and d_2 is the percentage of those who die per day. For simplicity, let $d = d_1 \cdot d_2$. (If you like, you could also denote $b(1-d_1)$ by, say, b'.

$$S' = cR - aSI$$

$$I' = aSI - b(1-d_1)I - dI$$

$$R' = b(1-d_1)I - cR$$

$$D' = dI$$

This Model Does Anything We Want!

- Note that these equations accommodate all the situations we have considered. To model an illness that doesn't kill anyone, just set $d_1 = 0$ (Hence $d = 0$). To model an illness for which recovery is permanent, just set $c = 0$. (If the illness lasts forever, then $b = 0$, and if it is not contagious, then $a = 0$, but we will not study those situations.)

Laboratory Exercise: Modeling Template

- Make a well organized spreadsheet that will calculate the populations of Susceptible, Infected, Recovered, and Dead people under any of the disease scenarios we have discussed.
- Your spreadsheet should include input cells for the following parameters:
  - a, the transmission coefficient
  - b, the recovery coefficient
  - c, the immunity loss coefficient
  - d_1 and d_2, the factors in the death coefficient
  - The time step Δt
  - The initial populations S, I, R, and D. (You may either include these in your list of parameters or just insert them into the initial cells of your calculation columns; that is a matter of taste.)

Laboratory Exercise: Calculation 1

- Input appropriate values for the parameters to model the following scenario:
  - No one dies of the illness, which lasts on average for 10 days. (Therefore, $D = 0$ initially and will remain 0.)
  - Immunity is permanent.
  - Initially there are $45,400$ susceptible people, 2100 infected people, and 2500 recovered people.
  - Immunity is permanent.
  - A typical person comes in contact with two out of every thousand other people in the population every day. (This is reasonable, if you think about it: 100 people per day. How do we know this?)
  - The infected population is evenly spread throughout the total population, and 1 contact in 20 results in transmission of the infection.

Laboratory Exercise: Calculation 1 (Continued)

- Calculate S, I, R, and D for the first 50 days using a time step of 1 day.
- Round all numbers to the nearest person.
- On a separate sheet of your workbook, graph S, I, R, and D as functions of time.
- Repeat the calculation on a third sheet using a time step of .1 day, and graph the results on a fourth sheet.
- Tip: It is helpful to label the sheets of your workbook!
- On two more sheets, repeat the calculation and graphs with a time step of .01 day. From your data, decide on an appropriate time step to get sufficiently accurate results.

What Went Wrong the First Time?

- If you used the parameters indicated by the description, you found that you got ridiculous numbers with $\Delta t = 1$ day. Some populations go negative while others blow up.
- Why did this happen?
- It happened because a time step of one day is too large for these parameters. The rates of change overshoot when extended over a full day without adjustment, leading to the crazy numbers.
- Remember that the rates in the model are changing all the time, adjusting to the changing populations in each category.
- Our spreadsheet is only an approximation of the model's behavior. If we don't recalculate the rates frequently enough, this approximation is very bad.

Laboratory Exercise: Calculation 2

The following scenario is somewhat similar to what is known about the Black Death plague that ravaged medieval Europe:

- The illness lasts on average for 15 days for those who recover, 5 days for those who die
- No treatment is available and the illness is very severe: 70% of those who contract the illness die.
- Immunity is permanent for those who recover.
- Since "contacts" are flea bites, the rate of infection is high: 1 contact in 3 results in infection.
- We'll leave the other assumptions the same:
  - Initially there are $45,400$ susceptible people, 2100 infected people, and 2500 recovered people.
  - A typical person comes in contact with two out of every thousand other people in the population every day (mostly via fleas biting people and rats).